• Spherical transformation of integrals

1. Compute
\[ \int_a^\rho \pi (\rho^2 - z^2) \, dz = \pi \left( \frac{2}{3} \rho^3 - a \rho^2 + \frac{1}{3} a^3 \right) \]

2. Let \( f(t) \) be a function on the interval \([a, b]\). What does the Mean Value Theorem say about \( f \)?

**Solution:** There is some \( c \in [a, b] \) such that \( f(b) - f(a) = f'(c)(b - a) \).

3. Let \( f(\rho) = \frac{1}{3} \rho^3 \) on the interval \([\rho_1, \rho_2]\). What does the mean value theorem say about this \( f \)?

**Solution:** There is some \( \rho \in [\rho_1, \rho_2] \) such that \( f(\rho_2) - f(\rho_1) = f'(\rho)(\rho_2 - \rho_1) \) or
\[ \frac{\rho_2^3}{3} - \frac{\rho_1^3}{3} = \rho^2(\rho_2 - \rho_1) \]

4. Let \( f(\phi) = \cos \phi \) on the interval \([\phi_1, \phi_2]\). What does the mean value theorem say about this \( f \)?

**Solution:** There is some \( \phi \in [\phi_1, \phi_2] \) such that \( f(\phi_2) - f(\phi_1) = f'(\phi)(\phi_2 - \phi_1) \) or
\[ \cos \phi_1 - \cos \phi_2 = \sin \phi (\phi_2 - \phi_1) \]

**Lecture Problems**

5.
\[ \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{x^2+y^2}^{\sqrt{18-x^2-y^2}} \left( x^2 + y^2 + z^2 \right) \, dz \, dx \, dy = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{3\sqrt{2}} \rho^4 \sin \phi \, d\rho \, d\theta \, d\phi = \frac{486\pi}{5} (\sqrt{2} - 1) \]

6. Let \( R \) be the region bounded by \( x^2 + y^2 = z^2 \) and \( x^2 + y^2 = 4y \). Set up the integral using spherical coordinates.
\[ \int\int\int_R \frac{1}{x^2 + y^2 + z^2} \, dV = \int_{\pi/4}^{3\pi/4} \int_0^{\pi/4} \int_0^{4\sin \theta / \sin \phi} \sin \phi \, d\rho \, d\theta \, d\phi = 4\pi \]