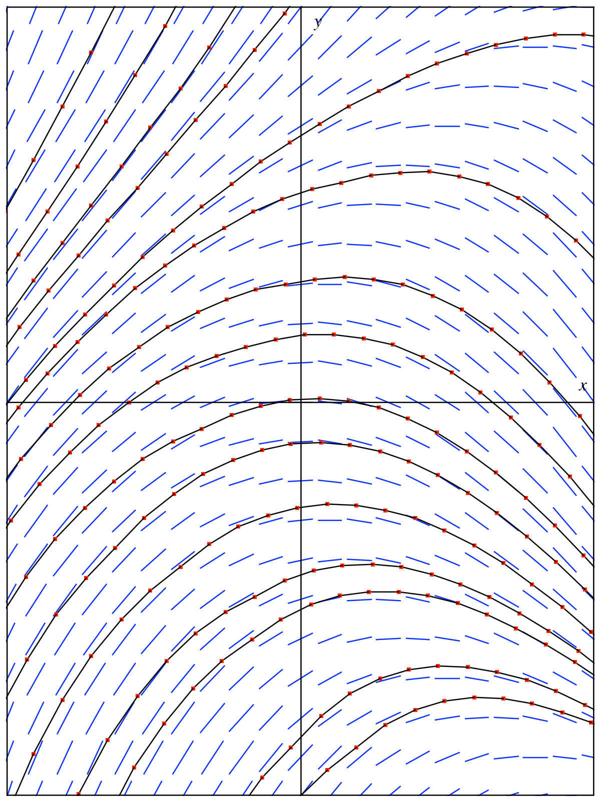
$\mathscr{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$	$\mathscr{L}\left\{t^{n}\right\} = \dfrac{n!}{s^{n+1}}$
$\mathscr{L}\left\{t^{a}\right\} = \dfrac{\Gamma(a+1)}{s^{a+1}}$	$\mathscr{L}\left\{\dfrac{1}{\sqrt{t}}\right\} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$
$\Gamma(x) = \displaystyle\int_0^\infty e^{-t}t^{x-1}\,dt$	$\Gamma(1/2) = \sqrt{\pi}$
$\mathscr{L}\left\{\cosh(at)\right\} = \dfrac{s}{s^2-a^2}$	$\mathscr{L}\left\{\sinh(at)\right\} = \dfrac{a}{s^2-a^2}$
$\mathscr{L}\left\{\cos(at)\right\} = \dfrac{s}{s^2+a^2}$	$\mathscr{L}\left\{\sin(at)\right\} = \dfrac{a}{s^2+a^2}$
$\mathscr{L}\left\{\dfrac{1}{2a^3}(\sin at - at\cos at)\right\} = \dfrac{1}{(s^2+a^2)^2}$	$\mathscr{L}\left\{\dfrac{t}{2a}\sin at\right\} = \dfrac{s}{(s^2+a^2)^2}$
$\mathscr{L}\left\{\dfrac{1}{2a}(\sin at + at\cos at)\right\} = \dfrac{s^2}{(s^2+a^2)^2}$	$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a)$
$\mathscr{L}\left\{f'(t)\right\} = s\,F(s) - f(0)$	$\mathscr{L}\left\{f''(t)\right\} = s^2\,F(s) - sf(0) - f'(0)$
$\mathscr{L}\left\{u_a(t)\right\} = \mathscr{L}\left\{u(t-a)\right\} = \dfrac{1}{s}e^{-sa}$	$\mathscr{L}\left\{u(t-a)f(t-a)\right\} = e^{-as}F(s)$
$\mathscr{L}\left\{\delta(t)\right\} = 1$	$\mathscr{L}\left\{\delta(t-a)\right\} = \mathscr{L}\left\{\delta_a(t)\right\} = e^{-sa}$
$\mathscr{L}\left\{\displaystyle\int_0^t f(u)\,du\right\} = \dfrac{1}{s}F(s)$	$\mathscr{L}^{-1}\left\{\dfrac{1}{s}F(s)\right\} = \displaystyle\int_0^t f(u)\,du$
$(f*g)(t) = \displaystyle\int_0^t f(u)g(t-u)\,du$	$\mathscr{L}\left\{(f*g)(t)\right\} = F(s)G(s)$
$\mathscr{L}\left\{t\,f(t)\right\} = -F'(s)$	$\mathscr{L}\left\{t^n\,f(t)\right\} = (-1)^nF^{(n)}(s)$
$\mathscr{L}\left\{\dfrac{1}{t}f(t)\right\} = \displaystyle\int_s^\infty F(u)\,du$	$\mathscr{L}^{-1}\left\{F(s)\right\} = t\mathscr{L}^{-1}\left\{\displaystyle\int_s^\infty F(u)\,du\right\}$
$\cosh t = \dfrac{1}{2}(e^t + e^{-t})$	$\sinh t = \dfrac{1}{2}(e^t - e^{-t})$

1. Find the differential equation that matches the given slope field



   (a) $y' = x^2 + y^2$
   (b) $y' = x - y^2$
   (c) $y' = x^2 - y^2$
   (d) $y' = x^2 - y$
   (e) $y' = y^2 - x \longleftarrow$ **CORRECT**
   (f) $y' = y^2 + x$

2. The equation below has exactly two equilibrium solutions, $x = a$ and $x = b$. Find a and b and determine the stability of these solutions.

   $$x' = (x - 5)(x^2 - 8x + 16)$$

   (a) $a + b = 9$ and both solutions are stable.
   (b) $a + b = 9$ and one solution is stable and the other is unstable.

   Solution: the equilibrium solutions are $x = 4, 5$, $x = 4$ is semi-stable and $x = 5$ is unstable

3. Solve

$$xy' = 2y + x^3 \cos x$$

Solution: $y = x^2 \sin x + Cx^2$

4. Suppose $y = \sum_{n=0}^{\infty} c_n x^n$ is a power series solution to

$$y'' + (\sin x)y = e^x, \qquad y(0) = 0, y'(0) = -1$$

What is c_4?

Solution: $c_4 = 1/8$.

5. The point $x = 0$ is a regular singular point to the equation below. Let a and b be the exponents of the of the differential equation at the singular point $x = 0$.

$$(x^2 - x)y'' + (\cos x + 1)y' + \frac{4}{x}y = 0$$

What is $a + b$?

Solution: $a = -1$ and $b = 4$ so $a + b = 3$

6. Consider the differential equation

$$(x-4)y'' + \frac{1}{x}y' + \frac{1}{\cos x}y = 0$$

Which of the following are true.

I. The point $x = 0$ is a regular singular point of the equation (with exponents . . .)
II. The point $x = 0$ is an irregular singular point of the equation
III. The point $x = 0$ is an ordinary point of the equation

Solution: I. the point is a regular singular point with exponents $r = 0, 5/4$.

7. Solve

$$(x^2 - 1)y'' - 6xy' + 12y = 0, \qquad y(0) = 1, y'(0) = 0$$

as a series solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

There was a typo on the original version of the problem, the second term should have a prime on the y.

Solution: The recurrence relation is $c_{n+2} = \frac{(n-3)(n-4)}{(n+2)(n+1)}c_n$. This forces $c_5 = c_6 = 0$ which makes $c_n = 0$ for $n \geq 5$. Because of the initial conditions the odd terms, $c_1 = c_3 = 0$. The solution is $y = 1 + 6x^2 + x^4$.

8. Consider

$$4xy'' + 2y' + y = 0.$$

This equation has $x = 0$ as a regular singular point with exponents 0 and $\frac{1}{2}$ (you don't need to check this).

I. There exists a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ (where the coefficients, a_n, can be determined).

II. There exists a solution of the form $y = \sqrt{x} \sum_{n=0}^{\infty} b_n x^n$ (where the coefficients, b_n, can be determined).

Solution: Both I. and II. are correct

9. Suppose $f(x)$ is a solution of $y' - 2y = x$, $y(0) = 1$. Find $\mathscr{L}\{f\}(3)$

Solution: The Laplace transform of the equation is $sY(s) - 1 - 2Y(s) = 1/s^2$. Setting $s = 3$ and solving we have $\mathscr{L}\{y\}(3) = Y(3) = 10/9$

10. Which of the following statements about the differential equation $ydx + xdy = 0$ are true?

(a) The ODE is exact $\longleftarrow$ **CORRECT**

(b) The ODE is separable $\longleftarrow$ **CORRECT**

(c) The ODE is first order linear $\longleftarrow$ **CORRECT**

(d) The ODE is homogeneous $\longleftarrow$ **CORRECT**

11. Consider the ODE $(y - x)y' = 1$. On which of the following rectangles does a unique solution exist?

(a) $(0, 1) \times (0, 1)$

(b) $(0, 2) \times (1, 2)$

(c) $(1, 2) \times (1, 2)$

(d) $(2, 3) \times (0, 1)$ $\longleftarrow$ **CORRECT**

(e) $(1, 3) \times (0, 2)$

Solution: Writing $y' = f(x, y) = 1/(y - x)$ the function f is not continuous along the line $y = x$. The only rectangle that does not touch this line is $(2, 3) \times (0, 1)$.

12. The autonomous equation $y' = (y - 4)^2(y + 2)(y - 3)(y - 5)$ has 4 equillibra points. Find the sum of the unstable ones.

(a) -2

(b) -1

(c) 3 $\longleftarrow$ **CORRECT**

(d) 5

(e) 7

(f) 8

(g) 10

13. Let

$$f(t) = \begin{cases} 3-t & t<4 \\ 0 & t=4 \\ t & t>4 \end{cases}$$

Compute $\mathscr{L}\{f\}(2)$.

Solution: Use the definition,

$$\begin{aligned} \mathscr{L}\{f\}(2) &= \int_0^\infty f(t)e^{-2t}\,dt = \int_0^4 (3-t)e^{-2t}\,dt + \int_4^\infty te^{-2t}\,dt \\ &= \frac{3}{4}e^{-8} + \frac{5}{4} + \frac{9}{4}e^{-8} = 3e^{-8} + \frac{5}{4} \end{aligned}$$

14. Find $\mathscr{L}\{\sin^2(t)\}$

Solution:

$$\mathscr{L}\{\sin^2(t)\} = \frac{1}{2}\mathscr{L}\{1-\cos(2t)\} = \frac{1}{2}\Big(\frac{1}{s} - \frac{s}{s^2+4}\Big) = \frac{2}{s(s^2+4)}$$

15. Which of the following 2nd order ODEs have ordinary points at $x = 0$?

(a) $y'' + xy' + (\sin x)y = 0 \longleftarrow$ **CORRECT**

(b) $y'' + xy' + (\cos x)y = 0 \longleftarrow$ **CORRECT**

(c) $xy'' + xy' + (\sin x)y = 0 \longleftarrow$ **CORRECT**

(d) $xy'' + xy' + (\cos x)y = 0$

16. Find a nontrivial lower bound for the radius of convergence of a power series solution of $(x^2 + 1)(x+1)y'' + xy' + y = 0$ centered at $x = 1$.

Solution: If ρ is the radius of convergence then it is as least as big as the distance from $x = 1$ to the nearest singular point, which in this case is $z = i$. Thus $\rho \geq \sqrt{2}$

17. What is the radius of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}x^n.$$

(a) $\rho = 0$

(b) $\rho = 1/4$

(c) $\rho = 1/2$

(d) $\rho = 1$

(e) $\rho = 2$

(f) $\rho = 4 \longleftarrow$ **CORRECT**

(g) $\rho = \infty$

18. Consider the differential equation

$$x^2(x-1)y'' + (\sin^2 x)y' + (x^4 + x^2)y = 0.$$

I. $x = 0$ is an ordinary point. $\longleftarrow$ **CORRECT**

II. $x = 0$ is an irregular singular point.

III. $x = 1$ is an ordinary point

IV. $x = 1$ is a regular singular point $\longleftarrow$ **CORRECT**

(a) I only

(b) II only

(c) III only

(d) IV only

(e) I and II

(f) I and III

(g) II and III

(h) III and IV

(i) I, II, and III

(j) I and IV $\longleftarrow$ **CORRECT** This was not an original choice.

19. (a) The Bessel function of order zero, denoted $J_0(t)$, is defined as a solution to differential equation

$$tx'' + x' + tx = 0 \qquad x(0) = 1, x'(0) = 0.$$

Show that a solution is given by the everywhere convergent power series

$$J_0(t) = \sum_{n=0}^{\infty} c_n t^n.$$

Solution: $t = 0$ is a regular singular point. The indicial equation is $r(r-1) + r = 0$ or $r^2 = 0$ so $r = 0$ is the exponent of the equation. The one Frobenius solution is

$$y = \sum_{n=0}^{\infty} c_n t^n$$

The recurrence relation is

$$c_n = -\frac{c_{n-2}}{n^2}$$

so

$$y = J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} t^{2n}.$$

The radius of convergence of this power series is $\rho = \infty$.

(b) Either use the differential equation or the power series to show that

$$\mathscr{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}},$$

Solution: The Laplace transform of the equation is

$$\mathscr{L}\{tx'' + x'tx\} = -(s^2+1)X'(s) - sX(s) = 0$$

which has the solution $X(s) = \frac{1}{\sqrt{s^2+1}}$. Or take the Laplace transform of the power series,

$$\begin{aligned}
\mathscr{L}\{J_0(t)\} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2}\mathscr{L}\{t^{2n}\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2}\frac{(2n)!}{s^{2n+1}} = \frac{1}{s}\sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}(s^{-2})^n \\
&= \frac{1}{s}\sum_{n=0}^{\infty} \frac{(1-2n)(3-2n)\cdots(-3)\cdot(-1)}{2^n n!}(s^{-2})^n \\
&= \frac{1}{s}\sum_{n=0}^{\infty} \frac{(\frac{1}{2}-n)(\frac{3}{2}-n)\cdots(-\frac{3}{2})\cdot(-\frac{1}{2})}{n!}(s^{-2})^n \\
&= \frac{1}{s}(1+s^{-2})^{-1/2} = \frac{1}{\sqrt{s^2+1}}
\end{aligned}$$

(c) Let $f(t) = (J_0 * \cos)(t)$, calculate

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\}.$$

Solution: First notice that $\mathscr{L}\{f(t)\} = \frac{s}{(s^2+1)^{3/2}}$ So

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \frac{\sigma}{(\sigma^2+1)^{3/2}}\, d\sigma = \frac{1}{\sqrt{s^2+1}}$$

20. Given that e^{2x} is a solution to the differential equation

$$y''' + y' - 10y = 0$$

find two other linearly independent solutions.

(a) $\{\cos x, \sin x\}$
(b) $\{e^x \cos x, e^x \sin x\}$
(c) $\{e^x \cos 2x, e^x \sin 2x\}$
(d) $\{e^{2x} \cos x, e^{2x} \sin x\}$
(e) $\{e^{-x} \cos x, e^{-x} \sin x\}$
(f) $\{e^{-x} \cos 2x, e^{-x} \sin 2x\} \longleftarrow$ **CORRECT**
(g) $\{e^{-2x} \cos x, e^{-2x} \sin x\}$

21. Express the general solution of $\dfrac{dy}{dx} = 1 + 2xy$ in terms of the error function

$$E(x) = \int_0^x e^{-t^2}\, dt.$$

(a) $y(x) = CE(x)$
(b) $y = Ce^{x^2} E(x)$
(c) $y = e^{x^2}(C + E(x))$
(d) $y = Ce^{-x^2} E(x)$
(e) $y = e^{-x^2}(C + E(x)) \longleftarrow$ **CORRECT**
(f) $y = e^{-x^2}(C + 2E(x))$

22. Suppose $y = \sum_{n=0}^{\infty} c_n x^n$ is the solution to

$$(x-3)(x+2)y'' + y' - 2x^5 y = 0$$

and $\rho = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|}$. Determine which inequality for ρ is guaranteed by the theorem discussed in class and the text book.

(a) $\rho \geq 1$
(b) $\rho \geq 2 \longleftarrow$ **CORRECT**
(c) $\rho \geq 3$
(d) $\rho \geq 4$
(e) $\rho \geq 5$
(f) $\rho \geq 6$

23. Consider the differential equation modelling a mass-spring system.

$$31x'' + 16x' + 2x = 0.$$

(a) This equation is an example of resonance
(b) This equation is an example of overdamping $\longleftarrow$ **CORRECT**
(c) This equation is an example of underdamping
(d) This equation is an example of critical damping
(e) None of the above

24. Suppose a mass-spring system with no damping has a mass of $m = 3$, spring constant of $k = 12$, and external force of $F_E(t) = -6\sin(\alpha t)$. What value of α will exhibit the phenomenon of resonance.

Solution: $\alpha = 2$

25. Find all eigenfunctions of the problem

$$y'' + \lambda y = 0, \qquad y'(0) = 0,\ y(\pi) = 0.$$

Solution: The eigenvalues are $\lambda_n = \frac{(2n-1)^2}{4}$ with eigenfunctions $y_n(x) = \cos\left(\frac{(2n-1)x}{2}\right)$

26. Find the Frobenius series solutions of

$$2xy'' + 3y' - y = 0.$$

Solution: The exponents are $r = 0, -1/2$ and the solutions are

$$y_1 = \sum_{n=0}^{\infty} \frac{x^n}{n!(2n+1)!!} = \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!} x^n.$$

and

$$y_2 = x^{-1/2} \sum_{n=0}^{\infty} \frac{x^n}{n!(2n-1)!!} = x^{-1/2} \sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^n$$