Name:
ID:
Section: ☐ Moen ☐ Thornton

This exam has:

- 14 multiple choice questions worth 5 points each.
- 2 hand graded questions worth 15 points each.

Important:

- No graphing calculators! Any non-graphing scientific calculator is fine.
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. You will be graded on the ease of reading your solution as well as for your work.
- You are allowed both sides of 3×5 "cheat card" for the exam.
- For the written section, please make a check next to your instructor on both pages.

$\mathscr{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$	$\mathscr{L}\left\{t^n\right\} = \dfrac{n!}{s^{n+1}}$
$\mathscr{L}\left\{t^a\right\} = \dfrac{\Gamma(a+1)}{s^{a+1}}$	$\mathscr{L}\left\{\dfrac{1}{\sqrt{t}}\right\} = \dfrac{\sqrt{\pi}}{\sqrt{s}}$
$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$	$\Gamma(1/2) = \sqrt{\pi}$
$\mathscr{L}\left\{\cosh(at)\right\} = \dfrac{s}{s^2-a^2}$	$\mathscr{L}\left\{\sinh(at)\right\} = \dfrac{a}{s^2-a^2}$
$\mathscr{L}\left\{\cos(at)\right\} = \dfrac{s}{s^2+a^2}$	$\mathscr{L}\left\{\sin(at)\right\} = \dfrac{a}{s^2+a^2}$
$\mathscr{L}\left\{\dfrac{1}{2a^3}(\sin at - at\cos at)\right\} = \dfrac{1}{(s^2+a^2)^2}$	$\mathscr{L}\left\{\dfrac{t}{2a}\sin at\right\} = \dfrac{s}{(s^2+a^2)^2}$
$\mathscr{L}\left\{\dfrac{1}{2a}(\sin at + at\cos at)\right\} = \dfrac{s^2}{(s^2+a^2)^2}$	$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a)$
$\mathscr{L}\left\{f'(t)\right\} = s\,F(s) - f(0)$	$\mathscr{L}\left\{f''(t)\right\} = s^2\,F(s) - sf(0) - f'(0)$
$\mathscr{L}\left\{u_a(t)\right\} = \mathscr{L}\left\{u(t-a)\right\} = \dfrac{1}{s}e^{-sa}$	$\mathscr{L}\left\{u(t-a)f(t-a)\right\} = e^{-as}F(s)$
$\mathscr{L}\left\{\delta(t)\right\} = 1$	$\mathscr{L}\left\{\delta(t-a)\right\} = \mathscr{L}\left\{\delta_a(t)\right\} = e^{-sa}$
$\mathscr{L}\left\{\int_0^t f(u)\,du\right\} = \dfrac{1}{s}F(s)$	$\mathscr{L}^{-1}\left\{\dfrac{1}{s}F(s)\right\} = \int_0^t f(u)\,du$
$(f*g)(t) = \int_0^t f(u)g(t-u)\,du$	$\mathscr{L}\left\{(f*g)(t)\right\} = F(s)G(s)$
$\mathscr{L}\left\{t\,f(t)\right\} = -F'(s)$	$\mathscr{L}\left\{t^n\,f(t)\right\} = (-1)^n F^{(n)}(s)$
$\mathscr{L}\left\{\dfrac{1}{t}f(t)\right\} = \int_s^\infty F(u)\,du$	$\mathscr{L}^{-1}\left\{F(s)\right\} = t\mathscr{L}^{-1}\left\{\int_s^\infty F(u)\,du\right\}$
$\cosh t = \dfrac{1}{2}(e^t + e^{-t})$	$\sinh t = \dfrac{1}{2}(e^t - e^{-t})$

1. Given the differential equation:

$$x^{(3)} + x'' + (\sin t)x' + x^2 + t^2 = e^{t^2}$$

select the system of differential equations that is equivalent to the equation above.

(a) $x_1' = x_2$
$x_2' = x_3 + (\sin t)x_2$
$x_3' = e^{t^2} - (x_1^2 + t^2)$

(b) $x_1' = x_1$
$x_2' = x_2$
$x_3' = e^{t^2} + (\sin t)x_2 - (x_1^2 + t^2)$

(c) $x_1' = x_1^2 + t^2$
$x_2' = (\sin t)x_2$
$x_3' = e^{t^2}$

(d) $x_1' = x_1^2 + t^2$
$x_2' = x_3 + (\sin t)x_1$
$x_3' = x_3$

(e) $x_1' = x_1$
$x_2' = x_2$
$x_3' = e^{t^2} - (x_1^2 + t^2) - (\sin t)x_2$

(f) $x_1' = x_2$
$x_2' = x_3$
$x_3' = e^{t^2} - (x_1^2 + t^2) - (\sin t)x_2 - x_3$ $\longrightarrow$ **CORRECT**

Solution: Use the method of introducing new variables $x_1 = x$, $x_2 = x' = x_1'$, $x_3 = x'' = x_2'$. This gives the equations

$$\begin{aligned} x_1' =& x_2 \\ x_2' =& x_3 \\ x_3' =& e^{t^2} - (x_1^2 + t^2) - x_2 \sin(t) - x_3 \end{aligned}$$

2. Find the inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{1}{\sqrt{s+1}}\right\}$$

(a) $\sqrt{\pi t}$

(b) $e^{-t}\sqrt{\pi t}$

(c) $\dfrac{e^{-t}}{\sqrt{\pi t}}$ $\longrightarrow$ **CORRECT**

(d) $e^{t}\sqrt{\pi t}$

(e) $e^{-t}\sqrt{\dfrac{\pi}{t}}$

(f) $e^{t}\sqrt{\dfrac{t}{\pi}}$

(g) $\dfrac{1}{\sqrt{\pi t}}$

(h) $\sqrt{\dfrac{\pi}{t}}$

(i) $\sqrt{\dfrac{t}{\pi}}$

Solution: Not that $\mathscr{L}^{-1}\{1/\sqrt{s}\} = \frac{1}{\sqrt{\pi t}}$. Thus, we use $\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)$ which gives $\mathscr{L}^{-1}\{1/\sqrt{s+1}\} = e^{-t}/\sqrt{\pi t}$.

3. Suppose y satisfies the initial value problem

$$ty'' + y' = -2, \qquad y(0) = 0, y'(0) = 0,$$

and $Y(s) = \mathscr{L}\{y(t)\}$. What differential equation does Y satisfy?

(a) $Y' + \frac{1}{s}Y = \frac{2}{s}$

(b) $Y' + \frac{1}{s}Y = \frac{2}{s^2}$

(c) $Y' + \frac{1}{s}Y = \frac{2}{s^3}$ $\longrightarrow$ **CORRECT**

(d) $Y' - \frac{1}{s}Y = \frac{2}{s^3}$

(e) $Y' + Y = \frac{2}{s^3}$

(f) $Y' + sY = 2$

(g) $Y' + sY = \frac{2}{s}$

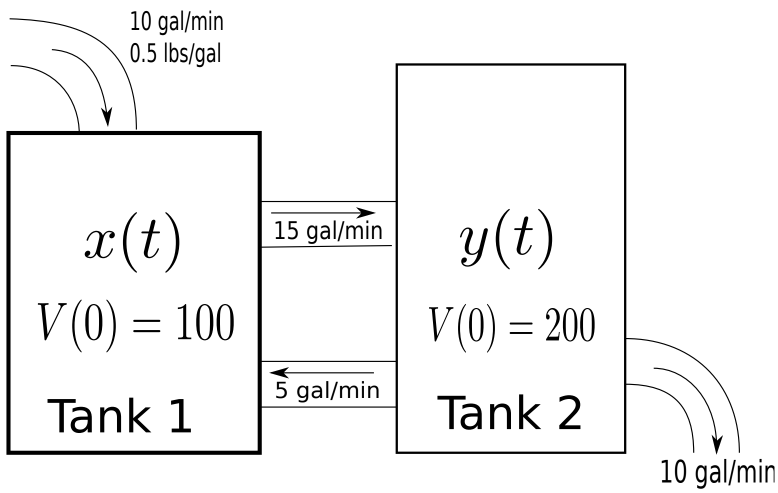
(h) $Y'' - \frac{1}{s}Y = \frac{2}{s^2}$

Solution:

$$\begin{aligned}
&\mathscr{L}\{ty''\} + \mathscr{L}\{y'\} = \mathscr{L}\{-2\} \\
&-\mathscr{L}\{Y''\}' + (sY - y(0)) = -\frac{2}{s} \\
&-[s^2Y - sy(0) - y'(0)]' + (sY - y(0)) = -\frac{2}{s} \\
&-(s^2Y)' + sY = -\frac{2}{s} \\
&-(s^2Y' + 2sY) + sY = -\frac{2}{s} \\
&-s^2Y' - sY = -\frac{2}{s}
\end{aligned}$$

4. Two tanks filled with brine are connected together as in the figure below. Tank 1, with $x(t)$ lbs of salt in it, starts with 100 gallons of brine. Tank 2, with $y(t)$ lbs of salt in it, starts with 200 gallons of brine. Brine at concentration 0.5 lbs/gal flows into Tank 1 at a rate of 10 gal/min. Brine is pumped from Tank 1 to Tank 2 at at rate of 15 gal/min. Brine is pumped from Tank 2 to Tank 1 at at rate of 5 gal/min. Brine flows from Tank 2 at a rate of 10 gal/min. As usual, we assume that the tanks are completely mixed.

   Which of the following systems of equations models this system?



   (a) $x' = -15x + 5y + 5$
   $y' = 15x - 5y$

   (b) $x' = -15x + 5y$
   $y' = 15x - 5y$

   (c) $40x' = -6x + y + 200$
   $40y' = 6x - 3y \longrightarrow$ **CORRECT**

   (d) $40x' = 5x - y + 5$
   $40y' = 15x - 5y$

   (e) $40x' = -15x + 5y + 5$
   $40y' = 15x - 5y$

   (f) $40x' = -15x + 5y + 5$
   $40y' = 15x - 5y$

   (g) $40x' = -6x + y + 5$
   $40y' = 6x - 3y$

Solution: The equations are set up as usual

$$x' = (10)(0.5) + \frac{5y}{200} - \frac{15x}{100}$$
$$y' = \frac{15x}{100} - \frac{5y}{200} - \frac{10y}{200}$$

Simplifying and rearranging gives

$$40x' = -6x + y + 200$$
$$40y' = 6x - 3y$$

5. Find the inverse Laplace transform of

$$\frac{s+8}{s^3+4s}$$

(a) $1 - \sin 2t + 2\cos 2t$

(b) $-1 + \frac{1}{2}\sin 2t + 2\cos 2t$

(c) $2 + \sin 2t + 2\cos 2t$

(d) $2 + \frac{1}{2}\sin 2t + 2\cos 2t$

(e) $2 + \frac{1}{2}\sin 2t - 2\cos 2t \quad \longrightarrow$ **CORRECT**

(f) $2 - \frac{1}{2}\sin 2t + 2\cos 2t$

(g) $2 - \frac{1}{2}\sin 2t - 2\cos 2t$

(h) $2 + \sin 2t + \frac{1}{2}\cos 2t$

(i) $2 + \sin 2t - \frac{1}{2}\cos 2t$

(j) $2 - \sin 2t + \frac{1}{2}\cos 2t$

(k) $2 - \sin 2t - \frac{1}{2}\cos 2t$

Solution: double check this one, its new

$$\begin{aligned} \frac{s+8}{s^3+4s} &= \frac{2}{s} + \frac{1-2s}{s^2+4} \\ &= \frac{2}{s} + \frac{1}{s^2+4} - \frac{2s}{s^2+4} \\ \mathscr{L}\left\{\frac{s+4}{s^3+4s}\right\} &= 2 + \frac{1}{2}\sin 2t - 2\cos 2t \end{aligned}$$

6. Suppose that y is a smooth function and $Y(s) = \mathscr{L}\{y(t)\}$ satisfies

$$\mathscr{L}\{y\}(3) = 1, \qquad y(0) = 2, \qquad \text{and} \qquad y'(0) = 1.$$

What is the value of $\mathscr{L}\{y''\}(3)$?

(a) 0
(b) 1
(c) 2 $\longrightarrow$ **CORRECT**
(d) 3
(e) 4
(f) 5

Solution:

$$\begin{aligned}
\mathscr{L}\{y''\} =& s^2Y - sy(0) - y'(0) \\
=& s^2Y - 2s - 1 \\
\mathscr{L}\{y''\}(3) =& 3^2Y(3) - 2(3) - 1 = 9(1) - 6 - 1 = 2
\end{aligned}$$

7. Suppose $\mathbf{A}$ is a 2×2 matrix that satisfies

$$\mathbf{A}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix} \qquad \text{and} \qquad \mathbf{A}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

Find the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(a) $\mathbf{x}(t) = c_1 \begin{bmatrix}2\\4\end{bmatrix} e^{t} + c_2 \begin{bmatrix}1\\-1\end{bmatrix} e^{-t}$

(b) $\mathbf{x}(t) = c_1 \begin{bmatrix}1\\2\end{bmatrix} e^{-t} + c_2 \begin{bmatrix}1\\-1\end{bmatrix} e^{2t}$

(c) $\mathbf{x}(t) = c_1 \begin{bmatrix}2\\4\end{bmatrix} e^{2t} + c_2 \begin{bmatrix}1\\-1\end{bmatrix} e^{-t}$ $\longrightarrow$ **CORRECT**

(d) $\mathbf{x}(t) = c_1 \begin{bmatrix}1\\2\end{bmatrix} e^{-2t} + c_2 \begin{bmatrix}1\\-1\end{bmatrix} e^{t}$

(e) $\mathbf{x}(t) = c_1 \begin{bmatrix}1\\2\end{bmatrix} e^{-2t} + c_2 \begin{bmatrix}1\\-1\end{bmatrix} e^{-t}$

(f) $\mathbf{x}(t) = c_1 \begin{bmatrix}1\\2\end{bmatrix} e^{t} + c_2 \begin{bmatrix}1\\-1\end{bmatrix} e^{2t}$

Solution: The information provided gives you the eigenvalues and the eigenvectors.

8. Suppose that $F(s) = \mathscr{L}\{f(t)\}$ and that

$$\mathscr{L}^{-1}\{F'(s)\} = \arctan(t).$$

What is $f(t)$?

(a) $f(t) = \dfrac{-1}{t(t^2+1)}$

(b) $f(t) = -t \arctan t$

(c) $f(t) = \displaystyle\int_0^t \arctan(\tau)\, d\tau$

(d) $f(t) = -\dfrac{1}{t}\arctan(t)$ $\longrightarrow$ **CORRECT**

(e) $f(t) = \dfrac{1}{t^2+1}$

(f) $f(t) = \dfrac{-t}{t^2+1}$

Solution: NEED A SAMPLE FOR SAMPLE EXAM
Since $\mathscr{L}\{-tf(t)\} = F'(s)$ we know that $-tf(t) = \arctan t$ and therefore $f(t) = -\frac{1}{t}\arctan t$.

9. Let $f(t)$ be given by

$$f(t) = \begin{cases} 3 & \text{if } 2 < t < 6 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $\mathscr{L}\{f(t)\}$.

(a) $\frac{1}{s}(e^{-2s} - e^{-6s})$

(b) $\frac{1}{s}(e^{-2} - e^{-6})$

(c) $\frac{1}{s}(e^{-6s} - e^{-2s})$

(d) $\frac{1}{s}(e^{-6} - e^{-2})$

(e) $\frac{3}{s}(e^{-2s} - e^{-6s})$ $\longrightarrow$ **CORRECT**

(f) $\frac{3}{s}(e^{-2} - e^{-6})$

(g) $\frac{3}{s}(e^{-6s} - e^{-2s})$

(h) $\frac{3}{s}(e^{-6} - e^{-2})$

Solution:

$$\begin{aligned} \mathscr{L}f(t) &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_2^6 e^{-st} 3\, dt \\ &= \frac{3}{s}(e^{-2s} - e^{-6s}) \end{aligned}$$

10. Let λ_1 and λ_2 be the eigenvalues of the matrix A.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

What is $|\lambda_1 - \lambda_2|$?

(a) 2
(b) 3
(c) 4 $\longrightarrow$ **CORRECT**
(d) 5
(e) 6
(f) 7
(g) 8
(h) 9
(i) 10

Solution: The eigenvalues are 1 and 5.

11. Let A be the product of the matrices:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} -3 & -1 & 2 \\ -1 & 0 & 7 \\ 2 & 1 & 0 \end{bmatrix}$$

Find the entry in the matrix A that is in the middle middle of the top row (row 1, column 2)?

(a) -4
(b) -3
(c) -2
(d) -1
(e) 0
(f) $1 \longrightarrow$ **CORRECT**
(g) 2
(h) 3
(i) 4

Solution:

$$\begin{bmatrix} 1 & 2 & 2 \\ -2 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} -3 & -1 & 2 \\ -1 & 0 & 7 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 16 \\ 8 & 3 & -4 \\ -2 & 1 & 23 \end{bmatrix}$$

12. Let $F(s)$ be the Laplace transform of $f(t) = 3\sin 2t - \cosh 3t$. What is $F(4)$? (Select the closest answer.)

    (a) -0.4202
    (b) -0.2714 $\longrightarrow$ **CORRECT**
    (c) -0.2435
    (d) -0.1056
    (e) -0.0453
    (f) 0
    (g) 0.0194
    (h) 0.3297

    Solution:

$$F(s) = \frac{6}{s^2+4} - \frac{s}{s^2-9}$$
$$F(4) = -\frac{19}{70} \approx -0.2714$$

13. Let $f(t)$ be the inverse Laplace transform of $F(s) = \dfrac{2s}{(s-1)^2+4}$. What is $f(2)$?

    (a) -18.23
    (b) -15.25 $\longrightarrow$ **CORRECT**
    (c) -11.93
    (d) -7.13
    (e) -0.745
    (f) 0
    (g) 3.68
    (h) 5.43
    (i) 7.12

    Solution:

$$\begin{aligned} F(s) =& \frac{2(s-1)}{(s-1)^2+4} + \frac{2}{(s-1)^2+4} \\ f(t) =& e^t(\sin 2t + 2\cos 2t) \\ f(2) =& e^2(\sin 4 + 2\cos 4) \approx -15.25 \end{aligned}$$

14. Solve the differential equation

$$x'' + 4x = e^t\sqrt{t}, \qquad x(0) = x'(0) = 0$$

(a) $x = \sin t$
(b) $x = \sin 2t$
(c) $x = (\sin t) * \left(e^t\sqrt{t}\right)$
(d) $x = (\sin 2t) * \left(e^t\sqrt{t}\right)$
(e) $x = \left(\frac{1}{2}\sin 2t\right) * \left(e^t\sqrt{t}\right)$ $\longrightarrow$ **CORRECT**
(f) $x = (2\sin 2t) * \left(e^t\sqrt{t}\right)$
(g) $x = (\cos t) * \left(e^t\sqrt{t}\right)$
(h) $x = (\cos 2t) * \left(e^t\sqrt{t}\right)$
(i) $x = \left(\frac{1}{2}\cos 2t\right) * \left(e^t\sqrt{t}\right)$
(j) $x = (2\cos 2t) * \left(e^t\sqrt{t}\right)$

Solution:

$$\begin{aligned} x'' + 4x &= e^t\sqrt{t} \\ s^2X + 4X &= \mathscr{L}\{e^t\sqrt{t}\} \\ X &= \frac{1}{s^2+4} \cdot \mathscr{L}\{e^t\sqrt{t}\} \\ x &= \left(\frac{1}{2}\sin 2t\right) * (e^t\sqrt{t}) \end{aligned}$$

WRITTEN PROBLEM—SHOW YOUR WORK

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Note: You will be graded on the readability of your work. Use the back of this sheet, if necessary.

15. Solve the following initial value problem.

$$x'' + 2x' - 15x = 8\,\delta(t-6) \qquad x(0) = 1, x'(0) = 0$$

Solution:

$$x'' + 2x' - 15y = 8\,\delta(t-6)$$
$$(s^2X - sx(0) - x'(0)) + 2(sX - x(0)) - 15X = 8e^{-6s}$$
$$s^2X - s + 2sX - 2 - 15X = 8e^{-6s}$$
$$(s^2 + 2s - 15)X = 8e^{-6s} + s + 2$$
$$X = \frac{8e^{-6s}}{(s-3)(s+5)} + \frac{s+2}{(s-3)(s+5)} = e^{-6s}\left(\frac{1}{s-3} - \frac{1}{s-5}\right) + \left(\frac{3}{8(s+5)} + \frac{5}{8(s-3)}\right)$$
$$x(t) = u(t-6)\left[e^{3(t-6)} - e^{-5(t-6)}\right] + \frac{3}{8}e^{-5t} + \frac{5}{8}e^{3t}$$

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Note: You will be graded on the readability of your work. Use the back of this sheet, if necessary.

16. Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Solution: The eigenvalue of the matrix are $2 \pm i$. Taking the eigenvalue $\lambda_1 = 2 + i$ and finding the eigenvectors gives the vector $v_1 = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$. Finding the real and complex parts of the solution $\mathbf{x} = v_1 e^{\lambda t}$ gives

$$\begin{aligned} v_1 e^{\lambda_1 t} &= \begin{bmatrix} 2 \\ 1+i \end{bmatrix} e^{(2+i)t} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix} e^{2t}(\cos t + i \sin t) \\ &= \begin{bmatrix} 2e^{2t}(i \sin t + \cos t) \\ (i+1)e^{2t}(\cos t + i \sin t) \end{bmatrix} \\ &= \begin{bmatrix} 2e^{2t} \cos t \\ e^{2t}(\cos t - \sin t) \end{bmatrix} + i \begin{bmatrix} 2e^{2t} \sin t \\ e^{2t}(\sin t + \cos t) \end{bmatrix} \end{aligned}$$

Thus, the general solution is

$$\mathbf{x} = c_1 e^{2t} \begin{bmatrix} 2 \cos t \\ \cos t - \sin t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \sin t \\ \sin t + \cos t \end{bmatrix}$$

Using the initial conditions gives

$$\begin{aligned} 2c_1 &= 2 \\ c_1 + c_2 &= 3 \end{aligned}$$

Therefore $c_1 = 1$ and $c_2 = 2$. Thus, the solution is

$$\mathbf{x} = e^{2t} \begin{bmatrix} 2 \cos t \\ \cos t - \sin t \end{bmatrix} + 2e^{2t} \begin{bmatrix} 2 \sin t \\ \sin t + \cos t \end{bmatrix}$$