1. Write the system
$$\begin{cases} x'' = y - 2x \\ y' = x + 2y \end{cases}$$
as a linear differential equation.

Solution:
$$x''' = y' - 2x' = x + 2y - 2x' = x + 2(x'' + 2x) - 2x' \Rightarrow x''' - 2x'' + 2x' - 5x = 0$$

2. Solve the initial value problem
$$\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 5 & -3 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Solution:
$$\mathbf{x}(t) = \begin{bmatrix} 7e^{2t} - 2e^{-2t} \\ 7e^{2t} - 10e^{-2t} \end{bmatrix}$$

3. Suppose $F(s) = \mathscr{L}\{\sqrt{t^2+1}\}$, calculate
$$\mathscr{L}^{-1}\Big\{\frac{F'(s)}{s}\Big\}.$$

Solution:
$$\mathscr{L}^{-1}\Big\{\frac{F'(s)}{s}\Big\} = -\int_0^t \tau\sqrt{\tau^2+1}\,d\tau = \frac{1-(t^2+1)^{3/2}}{3}.$$

4. Let $\mathbf{x}$ be the solution to
$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
What is $\mathbf{x}(\pi)$?

Solution:
$$\mathbf{x}(\pi) = \begin{bmatrix} (\cos\pi + \sin\pi)e^{\pi} \\ (\sin\pi - \cos\pi)e^{\pi} \end{bmatrix} = \begin{bmatrix} -e^{\pi} \\ e^{\pi} \end{bmatrix}$$

5. The Bessel function of order 0, denoted $J_0(t)$, can be define for $t \geq 0$ via the Laplace transform as
$$\mathscr{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}.$$
What is $(J_0 * J_0)(t)$?

Solution:
$$(J_0 * J_0)(t) = \sin t.$$

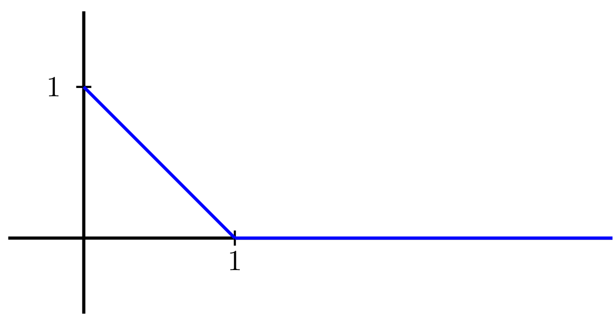
6. Solve the system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 5 \\ 2 & 1 & -2 \end{bmatrix} \mathbf{x}$$

Solution:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -8 \\ 19 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} e^{3t}.$$

7. Let $f(t)$ be given by the graph



Calculate $\mathscr{L}\{f(t)\}$, as well as its domain.

Solution: Domain: all real numbers,

$$\mathscr{L}\{f(t)\} = \begin{cases} \dfrac{e^{-s} + s - 1}{s^2} & s \neq 0 \\[2ex] \dfrac{1}{2} & s = 0 \end{cases}$$

8. The matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

has eigenvalues -1 and 3. Find the solution to the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$?

Solution:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}.$$

9. Use the Laplace transform to solve

$$y'' - 3y' + 2y = \sin t, \qquad y(0) = 1, y'(0) = 2$$

Solution: Taking the Laplace transform

$$(s^2 - 3s + 2)Y(s) - s + 1 = \frac{1}{s^2 + 1}$$

$\Rightarrow$

$$Y(s) = \frac{1}{(s-1)(s-2)(s^2+1)} + \frac{s-1}{(s-2)(s-1)} = \frac{6}{5}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s-1} + \frac{3}{10}\frac{s}{s^2+1} + \frac{1}{10}\frac{1}{s^2+1}.$$

$\Rightarrow$

$$y(t) = \frac{6}{5}e^{2t} - \frac{1}{2}e^t + \frac{3}{10}\cos t + \frac{1}{10}\sin t.$$

10. Solve the differential equation

$$tx'' - 2x' + tx = -\cos t, \qquad x(0) = x'(0) = 0.$$

(Hint: If $X(s) = \mathscr{L}\{x(t)\}$ then X satisfies a linear first order differential equation).

Solution: Taking the Laplace transform and simplifying

$$(s^2+1)X'(s) + 4sX(s) = \frac{s}{s^2+1}$$

Multiply by the integrating factor $\Rightarrow$

$$(s^2+1)^2X'(s) + 4s(s^2+1)X(s) = s$$

$\Rightarrow$

$$(s^2+1)^2X(s) = \frac{s^2}{2} + C$$

$\Rightarrow$

$$X(s) = \frac{s^2}{2(s^2+1)^2} + \frac{C}{(s^2+1)^2}.$$

$\Rightarrow$

$$x(t) = \frac{1}{4}(\sin t + t\cos t) + \frac{C}{2}(\sin t - t\cos t).$$

11. Solve the system

$$\begin{aligned} x_1' &= 2x_1 - x_2 \\ x_2' &= 3x_1 + x_2 \end{aligned}$$

Solution: The eigenvalues are $\lambda = \frac{3}{2} \pm \frac{\sqrt{11}}{2}i$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2\cos(\frac{\sqrt{11}}{2}t) \\ \cos(\frac{\sqrt{11}}{2}t) + \sqrt{11}\sin(\frac{\sqrt{11}}{2}t) \end{bmatrix} e^{\frac{3}{2}t} + c_2 \begin{bmatrix} -2\sin(\frac{\sqrt{11}}{2}t) \\ -\sin(\frac{\sqrt{11}}{2}t) + \sqrt{11}\cos(\frac{\sqrt{11}}{2}t) \end{bmatrix} e^{\frac{3}{2}t}.$$

Note: answers may vary depending on choice of eigenvalues and eigenvectors.

12. If x and y are solutions of $x' = x + 3y$, $y' = x + 4y$, $x(0) = 2$, $y(0) = 1$, find $x(1)$.

Solution: The eigenvalues are $\lambda = \frac{5}{2} \pm \frac{\sqrt{21}}{2}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{(5-\sqrt{21})t/2} \begin{bmatrix} 1 \\ -\frac{3-\sqrt{21}}{6} \end{bmatrix} + e^{(5+\sqrt{21})t/2} \begin{bmatrix} 1 \\ \frac{3+\sqrt{21}}{6} \end{bmatrix}$$

13. Find the product of the eigenvalues of $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & -2 \end{bmatrix}$

    Solution: The eigenvalues are $\lambda = 1, \pm\sqrt{6}$ so the product is -6.

14. Compute $\mathscr{L}\left\{\dfrac{e^t - \cos t}{t}\right\}$.

    Solution:

$$\mathscr{L}\left\{\frac{e^t - \cos t}{t}\right\} = \int_s^\infty \frac{1}{\sigma - 1} - \frac{\sigma}{\sigma^2 + 1}\, d\sigma = \ln\left(\frac{\sqrt{s^2+1}}{s-1}\right), \qquad s > 1.$$

15. Let $f(t) = \mathscr{L}^{-1}\left\{\dfrac{-3s^2 + 3s + 4}{s^3 + 4s^2}\right\}$. Compute $f(0)$.

    Solution:

$$\mathscr{L}^{-1}\left\{\frac{-3s^2 + 3s + 4}{s^3 + 4s^2}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{2s} + \frac{1}{s^2} - \frac{7}{4}\frac{1}{s+4}\right\} = \frac{1}{2} + t - \frac{7}{4}e^{-4t}$$

    so $f(0) = -5/4$.

16. A mass $m = 1$ is attached to a spring with constant $k = 4$. Assume no damping force is present and the mass lie at rest in its equilibrium position. At time $t = 0$, the mass is hit with a hammer with a unit impulse in the positive direction.

    (a) $x(t) = \frac{1}{2}\sin(2t)$

    (b) $t = 0, \pi/2, \pi, 3\pi/2, 2\pi, 5\pi/2, 3\pi$

    At time $t = 3\pi$, the mass is once again struck with a unit impulse in the positive direction.

    (c) $t = 7\pi/2, 4\pi, 9\pi/2, 5\pi, \ldots, (2n-1)\pi/2, n\pi, \ldots,\ n = 4, 5, \ldots$

    (d)