Name:
ID:
Section: ☐ Moen ☐ Thornton

This exam has 17 questions:

- 14 multiple choice questions worth 5 points each.
- 2 hand graded questions worth 15 points each.

Important:

- No graphing calculators! Any non-graphing scientific calculator is fine.
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. You will be graded on the ease of reading your solution as well as for your work.
- You are allowed both sides of 3×5 "cheat card" for the exam.
- For the written section, please make a check next to your instructor on both pages.

1. Which of the following can be solved by the method of undetermined coefficients?

   I. $y'' - y = x^{100}$
   II. $y'' - y = x^{-100}$
   III. $y'' - y = x^{100}e^{x}$
   IV. $y'' - y = x^{-100}e^{x}$

   (a) I only
   (b) II only
   (c) III only
   (d) I and II only
   (e) I and III only $\longrightarrow$ **CORRECT**
   (f) III and IV only
   (g) I, II, and III
   (h) All can be solved by the method of undetermined coefficients
   (i) None can be solved by the method of undetermined coefficients

2. If y is a solution of the IVP

$$y'' + 2y' + 2y = 0, \quad y(0) = 2, y'(0) = -3$$

Find $y(\pi/4)$. Pick the closest answer.

(a) 0
(b) 1.571
(c) 0.707
(d) 2.257
(e) 0.967
(f) -0.967
(g) 0.322 $\longrightarrow$ **CORRECT**
(h) -0.322

Solution:

$$\begin{aligned} y =& e^{-x}(c_1 \cos x + c_2 \sin x) \\ y =& e^{-x}(2\cos x - \sin x) \\ y(\pi/4) \approx& 0.3223969 \end{aligned}$$

3. Which of the following sets of functions are linearly independent on the real line?
Recall: $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

I. $\{\sin(x), \cos(x), e^x\}$
II. $\{\sinh(x), \cosh(x), e^x\}$
III. $\{\sin(x), \cosh(x), e^x\}$

(a) I only
(b) II only
(c) III only
(d) I and II only
(e) I and III only $\longrightarrow$ **CORRECT**
(f) III and IV only
(g) I, II, and III
(h) None of the sets are linearly independent.

Solution:

$$\begin{aligned} W(\sin(x), \cos(x), e^x) =& -2e^x \neq 0 \\ W(\sinh(x), \cosh(x), e^x) =& 0 \\ W(\sin(x), \cosh(x), e^x) =& 2e^x \sin x(\sinh x - \cos x) \neq 0 \end{aligned}$$

4. Which of the following equations have $e^x \sin 2x$ as a solution?

   I. $y^{(3)} - 2y'' + 5y' = 0$
   II. $y'' + 2y' + 5y = 0$
   III. $2y'' - 4y' + 10y = 0$

   (a) I only
   (b) II only
   (c) III only
   (d) I and II only
   (e) I and III only $\longrightarrow$ **CORRECT**
   (f) II and III only
   (g) I, II, and III

   Solution: Factor the characteristic equations:

$$r^3 - 2r^2 + 5r = r[r - (1 + 2i)][r - (1 - 2i)]$$
$$r^2 + 2r + 5 = [r - (-1 + 2i)][r - (-1 - 2i)]$$
$$2r^2 - 4r + 10r = 2[r - (1 + 2i)][r - (1 - 2i)]$$

5. Given the boundary value problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, y(\pi) = 0,$$

   what value of λ will provide a non-trivial solution (i.e. a solution other than $y \equiv 0$)?

   (a) $\lambda = -4$
   (b) $\lambda = -1$
   (c) $\lambda = 0$
   (d) $\lambda = 2$
   (e) $\lambda = 3$
   (f) $\lambda = 4$ $\longrightarrow$ **CORRECT**
   (g) $\lambda = \pi$
   (h) $\lambda = \pi^2$

   Solution: If *lambda* < 0 then the general solution is $Ae^{x\sqrt{\lambda}} + Be^{x\sqrt{\lambda}}$. Using the boundary conditions gives $A = B = 0$.

   If *lambda* $= 0$ then the general solution is $Ax + B$. Using the boundary conditions gives $A = B = 0$.

   If *lambda* > 0 then the general solution is $A\cos(t\sqrt{\lambda}) + B\sin(t\sqrt{\lambda})$. Using the boundary conditions gives $A = 0$ and $B \sin\sqrt{\lambda} = 0$. This gives $\sqrt{\lambda} = 1, 2, 3, \ldots$. Thus, the eigen values are $\lambda = 1, 4, 9, 16, \ldots$.

6. A mass weighing 50 kg is attached to the end of a spring that is stretched 0.1 m by a force of 10 N. An external force $F = 20\cos(\omega t)$ also acts on the mass. At what value of ω will resonance occur? Neglect any friction or damping.

   (a) $\omega = 0$
   (b) $\omega = 1$
   (c) $\omega = \sqrt{2}$ $\longrightarrow$ **CORRECT**
   (d) $\omega = 2$
   (e) $\omega = 2\sqrt{2}$
   (f) $\omega = 4$
   (g) $\omega = 6$

   Solution: $50x'' + 100x = 20\cos\omega t$. The natural frequency of the system is $\omega_0 = \sqrt{k/m} = \sqrt{2}$. This is when resonance occurs.

7. Find the general solution to

   $$y''' + 6y'' + 9y' = 0$$

   (a) $c_1 + c_2e^{-3x} + c_3xe^{-3x}$ $\longrightarrow$ **CORRECT**
   (b) $c_1 + c_2e^{-3x} + c_3e^{3x}$
   (c) $c_1 + c_2x + c_3e^{-3x}$
   (d) $c_1 + c_2e^{3x} + c_3xe^{3x}$
   (e) $c_1x + c_2e^{-3x} + c_3xe^{-3x}$
   (f) $c_1 + c_2xe^{-3x} + c_3e^{3x}$
   (g) $c_1 + c_2e^{-3x} + c_3xe^{3x}$

   Solution: The characteristic equation is

   $$r^3 + 6r^2 + 9r = r(r+3)^2 = 0$$

8. Solve the initial value problem and find $y(\pi/2)$.

$$y'' + y = 6\sin(2x), \qquad y(0) = 1, y'(0) = 2$$

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4
(f) 5
(g) 6 $\longrightarrow$ **CORRECT**
(h) 7

Solution:

$$\begin{aligned} y_c =& c_1 \cos x + c_2 \sin x \\ y_p =& -2\sin 2x \\ y_{gen} =& c_1 \cos x + c_2 \sin x - 2\sin 2x \\ y =& \cos x + 6\sin x - 2\sin 2x \\ y(\pi/2) =& 6 \end{aligned}$$

9. Find the Wronskian of $e^{2x}\cos(x)$ and $e^{2x}\sin(x)$.

(a) e^{2x}
(b) $2e^{2x}$
(c) $e^{2x}\sin(x)\cos(x)$
(d) $2e^{2x}\sin(x)\cos(x)$
(e) e^{4x} $\longrightarrow$ **CORRECT**
(f) $2e^{4x}$
(g) $e^{4x}\sin(x)\cos(x)$
(h) $2e^{4x}\sin(x)\cos(x)$

10. Suppose you were to use the method of undetermined coefficients to solve the differential equation

$$y''' + y' = x + \sin(x).$$

As discussed in class and the textbook, what is the best correct form for a particular solution.

(a) $y_p = Ax + B\sin(x)$
(b) $y_p = A + Bx + C\sin(x) + D\cos(x)$
(c) $y_p = Ax + Bx^2 + C\sin(x) + D\cos(x)$
(d) $y_p = A + Bx + Cx\sin(x) + Dx\cos(x)$
(e) $y_p = Ax + Bx^2 + Cx\sin(x) + Dx\cos(x)$ $\longrightarrow$ **CORRECT**
(f) $y_p = Ax^2 + Bx^3 + Cx\sin(x) + Dx\cos(x)$
(g) $y_p = Ax + Bx^2 + Cx^2\sin(x) + Dx^2\cos(x)$
(h) $y_p = Ax^2 + Bx^3 + Cx^2\sin(x) + Dx^2\cos(x)$

11. Suppose $y = e^{-x}\sin 2x$ is a solution to the constant coefficient homogeneous differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $a_n, \ldots, a_0$ are real numbers. Which of the following must be true?

(a) $r = 2 - i$ is a root of the characteristic equation.
(b) $y = e^{x}\sin 2x$ is a solution to the differential equation.
(c) $y = \cos 2x$ is a solution to the differential equation.
(d) $r = 1 + 2i$ is a root of the characteristic equation.
(e) $y = e^{2x}\cos x$ is a solution of the differential equation.
(f) $y = e^{-x}\cos 2x$ is a solution of the differential equation. $\longrightarrow$ **CORRECT**

12. Suppose y_1 and y_2 are particular solutions of

$$y''' - 4y'' + 3y' = f(x)$$

where $f(x)$ is a non zero differentiable function.
Which of the following are also solutions to the above differential equation?

I. $y(x) = 1 + y_1(x)$
II. $y(x) = 2y_1(x) - y_2(x)$
III. $y(x) = 1 + e^x + e^{3x}$
IV. $y(x) = e^x - y_2(x)$

(a) I only
(b) II only
(c) III only
(d) IV only
(e) I and II only $\longrightarrow$ **CORRECT**
(f) I and IV only
(g) II and III only
(h) I, II, and III only
(i) None of the above
(j) All of the above

13. Which of the following functions is a particular solution of

$$y''' - 4y' = 8t + 15\cos t + 8e^{-2t}?$$

(a) t
(b) e^{-2t}
(c) te^{-2t}
(d) $-3\sin t$
(e) $-t^2$
(f) $-t^2 - 3\sin t + te^{-2t}$ $\longrightarrow$ **CORRECT**

Solution: The correct answer is the only answer with the correct form. Just check that it is a solution.

14. Which of the following is **not** a solution to the differential equation

$$y^{(5)} + 8y^{(3)} + 16y' = 0$$

(a) $y = \sin(2x)$
(b) $y = -1$
(c) $y = 1 - 2x\sin(2x)$
(d) $y = x\cos(2x) - 2\sin(2x)$
(e) $y = 2 + \sin(2x) + \cos(2x)$
(f) $y = x + \sin(2x)$ $\longrightarrow$ **CORRECT**

Solution: Use the factorization

$$r^5 + 8r^3 + 16r = r(r^2+4)^2$$

Thus the general solution is

$$y = c_1 + c_2\cos 2x + c_3\sin 2x + c_4 x\cos 2x + c_5 x\sin 2x$$

WRITTEN PROBLEM—SHOW YOUR WORK

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Note: You will be graded on the readability of your work. Use the back of this sheet, if necessary.

15. (a) Show that $y_1(t) = e^t$ and $y_2(t) = t + 1$ are solutions to

$$y'' - \left(1 + \frac{1}{t}\right) y' + \frac{1}{t}\, y = 0$$

(b) Use the variation of parameters method to find the general solution to

$$y'' - \left(1 + \frac{1}{t}\right) y' + \frac{1}{t}\, y = t$$

Solution:

$$\begin{aligned}
W(y_1, y_2) =& - te^t \\
u_1' =& (t+1)e^{-t} \\
u_2' =& -1 \\
u_1 =& -(t+2)e^{-t} \\
u_2 =& -t \\
y_p =& -(t+2)e^{-t}e^t + (-t)(t+1) = -t^2 - 2t - 2 \\
y =& c_1 e^t + c_2(t+1) - t^2 - 2t - 2
\end{aligned}$$

WRITTEN PROBLEM—SHOW YOUR WORK

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16. (a) Find the general solution to

$$x'' + 4x' + 5x = 0.$$

(b) Consider a mass-spring-dashpot system with a mass of 1 kg, a spring with constant $k = 5$, and the dashpot providing 4 N of resistance for each meter per second of velocity. Is the system overdamped, critically damped, or underdamped?

Solution: Underdamped. The general solution to the homogeneous equation is

$$x = e^{-2t}(c_1 \cos(t) + c_2 \sin(t))$$

(c) Now suppose an external force acts on the mass-spring-dashpot system from part (b) providing an external force of $F_E(t) = 5\cos(2t)$, thus giving the differential equation

$$x'' + 4x' + 5x = 5\cos(2t)$$

describing the system. Find the general solution to this differential equation.

Solution:

$$x = e^{-2t}(c_1 \cos(t) + c_2 \sin(t)) + \frac{1}{13}\cos(2t) + \frac{8}{13}\sin(2t)$$