Important:

- No graphing calculators! Any non-graphing scientific calculator is fine.
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. You will be graded on the ease of reading your solution as well as for your work.
- You are allowed both sides of 3×5 "cheat card" for the exam.
- For the written section, please make a check next to your instructor on both pages.

1. Find a particular solution of

$$y'' + y = \sec x$$

Solution:

$$y_p = \frac{1}{2} \cos x \ln((\cos 2x + 1)/2) + x \sin x$$

2. Given that x^2 and x^3 are solutions of

$$x^2 y'' - 4xy' + 6y = 0$$

solve

$$x^2 y'' - 4xy' + 6y = x^4 e^x$$

Solution: $y = c_1 c^2 + c_2 c^3 + x^2 e^x$

3. If y is a solution fo the IVP

$$y''' - y'' - y' + y = 0, \qquad y(0) = 0, y'(0) = 4, y''(0) = 4$$

Find $y(1)$.

Solution: Note that $r^3 - r^2 - r + 1 = (r-1)^2(r+1)$.

4. A mass of 3 kg is suspended from a spring. It is pulled down 0.5m by a force of $\frac{\pi^2}{12}$N and released at time $t = 0$. What is the first time it returns to its initial position?

Solution: $k = \pi^2/6$, $t = 3\sqrt{2}/2$.

5. A mass of 2 kg is suspended from a spring. It is pulled down 1 m by a force of 4N, and undergoes free damped motion with a damping constant of 4 N/m/s. What is the farthest the mass ever is from its initial position?

Solution: $2x'' + 4x' + 4x = 0$. Initial position is $x(0) = 1$, and $x'(0) = 0$. Solving gives $x = e^{-t}(\cos t + \sin t)$. The furthest away is when $t = \pi$, so the furthest away is when $x = x(\pi) = e^{-pi}$ and the distance for the initial position is $1 + e^{-\pi}$.

6. If y is a solution of

$$y'' - 2y' = 3, \qquad y(0) = 4, y'(0) = \frac{11}{2}$$

find $y(2)$.

Solution: $y = \frac{7\,e^{2\,x}}{2} - \frac{6\,x+3}{4} + \frac{1}{4}$

7. Recall that a pendulum can be modeled by $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$, where g is a constant depending on the altitude. If a pendulum of length 5 in is at an altitude such that the period is 1s, how long would the pendulum need to be to have a period of 4s?

   Solution: First solve the equations, then use the information to find g. Then, using the g you found, find L.

8. Find a differential equation with $e^{3x}\sin 4x$ as one solution.

   Solution: One answer is $y'' - 6y' + 25 = 0$

9. Solve the initial value problem and find $y(1)$

$$y'' - 4y' + 13y = 0, \qquad y(0) = 1, y'(0) = 0$$

   Solution: $y = e^{2\,x}\left(cos\left(3\,x\right) - \frac{2\,sin(3\,x)}{3}\right)$

10. Suppose y_1 and y_2 are continuously differentiable functions on the real line and $W(y_1, y_2)(x) = x^4$, where W is the Wronskian. Which of the following are true.

    I) y_1 and y_2 are the solutions to the second-order linear differential equation

$$y'' + xy' + \sin(x)y = 0$$

    on the real line.

    II) y_1 and y_2 are linearly independent on the real line.

    III) $W(y_2, y_1) = -x^4$.

    (a) Only I
    (b) Only II
    (c) Only III
    (d) I and II
    (e) I and II
    (f) II and III
    (g) I, II, and III
    (h) None of the above

    Solution: Correct answer is f.

11. Are the functions below linearly independent?

$$\cos^2 3x, \sin^2 3x, x$$

    Solution: Yes

12. Consider

$$t^3y^{(5)} - \sin t y^{(3)} + e^t y' - y = \frac{e^{t^2}}{t-1}$$

    Suppose y_c is a solution to the corresponding homogeneous equation and y_{p_1} and y_{p_2} are both solutions to the nonhomogeneous equation. Which of the following are also solutions to the nonhomogeneous equation?

I. $y_c + y_{p_1}$ Yes
II. $y_{p_1} + y_{p_2}$ No
III. $y_{p_1} - y_{p_2}$ No
IV. $2y_{p_1} + y_{p_2}$ No
V. $2y_{p_1} - y_{p_2}$ Yes

Solution:

13. Find an equation with solution $y = 4e^x + 5e^{2x}\sin 3x$.

    Solution: One answer: $y''' - 5y'' + 17y' - 13y = 0$.

14. (a) Find the general solution to

    $$y^{(3)} - 6y'' + 13y' - 10y = 0$$

    Hint: e^{2x} is a solution to the equation.
    Solution: Note that $r^3 - 6r^2 + 13r - 10 = (r-2)(r^2 - 4r + 5)$

    (b) Find a particular solution to (same eqn but nonhomogeneous)

    $$y^{(3)} - 6y'' + 13y' - 10y = 3e^x$$

    (c) Find the general solution to

    $$y^{(3)} - 6y'' + 13y' - 10y = 3e^x$$

    Solution:

15. Solve the initial value problem and find $y(2)$.

    $$y'' - 6y' + 9y = e^{3x}, \qquad y(0) = 0, y'(0) = 0$$

    Solution: $y = \frac{1}{2}x^2e^{3x}$

16. Which of the following are linearly independent solutions to

    $$(D-1)(D-2)^2(D^2 - 2D + 2)y = 0.$$

    (a) $e^x, e^{2x}, xe^{2x}, \sin(2x), \cos(2x)$
    (b) $e^x, xe^x, e^{2x}, e^{2x}\sin(2x), \cos(2x)$
    (c) $e^x, e^{2x}, xe^{2x}, e^x\sin(2x), e^x\cos(2x)$
    (d) $e^x, e^{2x}, xe^{2x}, e^x\sin(x), e^x\sin(x)$
    (e) $e^x, e^{2x}, xe^{2x}, e^{-x}\sin(x), e^{-x}\cos(x)$
    (f) $e^x, e^{2x}, xe^{2x}, e^{2x}\sin(x), e^{2x}\cos(x)$
    (g) $e^x, e^{2x}, xe^{2x}, e^x\sin(x), e^x\cos(x)$ $\longrightarrow$ **CORRECT**
    (h) $e^x, xe^x, e^{2x}, e^x\sin(x), e^x\cos(x)$

Solution:

17. NOTE: This material will not be on Exam 2!

    Suppose that a motorboat is moving at 50 ft/s when its motor is turned off and that 10 s later the boat has slowed to 25 ft/s. Assume that the resistance the boat encounters while coasting is proportional to its velocity. How far will the boat coast in all?

    (a) 503.1 ft
    (b) 652.6 ft
    (c) 721.3 ft
    (d) 1000 ft
    (e) 256.1 ft
    (f) 14.4 ft

    Solution:

18. Which of the following sets of functions is linearly independent on the real line?

    (a) $1, \cos^2(2x), \sin^2(2x)$
    (b) $x^2 - x,\ x^2,\ x$
    (c) $0, 1, x$
    (d) $x, \sin(x), \cos(x)$ $\longrightarrow$ **CORRECT**
    (e) $x, -x, |x|$
    (f) $0, x^2, \tan(x)$

    Solution:

19. In a mass-spring-dashpot system (i.e. free damped motion), which of the following are possible values for the total number of times the mass is at the equillibrium position?

    I. 0
    II. 1
    III. 2
    IV. ∞

    (a) I only
    (b) II only
    (c) III only
    (d) IV only
    (e) I and II
    (f) I and III
    (g) II and III
    (h) I, II, and III

(i) I, II and IV $\longrightarrow$ **CORRECT**

(j) II,III and IV

(k) I, III, and IV