Name:
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Section:

This exam has 16 questions:

- 14 multiple choice questions worth 5 points each.
- 2 hand graded questions worth 15 points each.

Important:

- No graphing calculators! Any non-graphing scientific calculator is fine.
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. You will be graded on the ease of reading your solution as well as for your work.
- You are allowed both sides of 3×5 note "cheat card" for the exam.

1. Consider the initial value problem

$$\frac{dy}{dx} = 2xy + 1 \qquad y(0) = 1$$

Approximate $y(1)$ using Euler's method with step size $h = 0.5$

(a) 0.25
(b) 1.0
(c) 1.5
(d) 1.75
(e) 2.0
(f) 2.25
(g) 2.5
(h) 2.75 $\longrightarrow$ **CORRECT**
(i) 3.0
(j) 3.25
(k) None of the above

Solution:

n	x_n	y_n	$f(x_n, y_x)$
0	0.0	1.0	1.0
1	0.5	1.5	2.5
2	1.0	2.75	

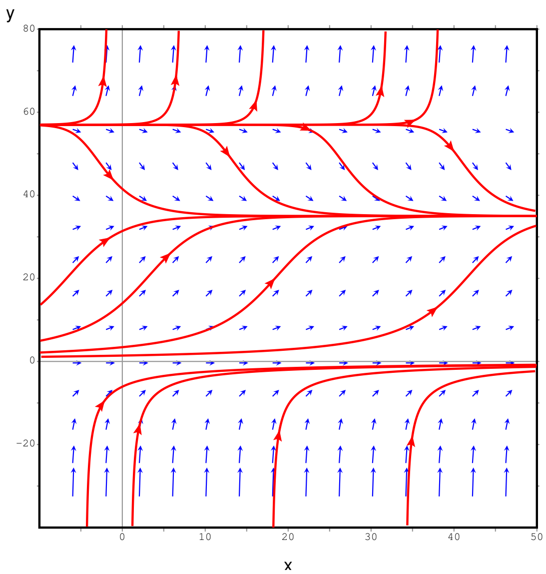
2. A small population of rabbits, $R(t)$, is modeled by the differential equation

$$\frac{dR}{dt} = \frac{1}{10^5} R^2 (R-35)(R-57), \qquad R(0) = 50$$

Determine $\lim_{t \to \infty} R(t)$.

(a) 0
(b) 35 $\longrightarrow$ **CORRECT**
(c) 50
(d) 57
(e) 1995
(f) 10^5
(g) ∞
(h) None of the above

Solution: You can see the solution if you look at the slopefield and some solution curves:



3. If y is a solution to

$$\frac{dy}{dx} = x(2-y)^2, \quad y(1) = 2$$

what is $y(2)$?

(a) -3

(b) -2

(c) -1

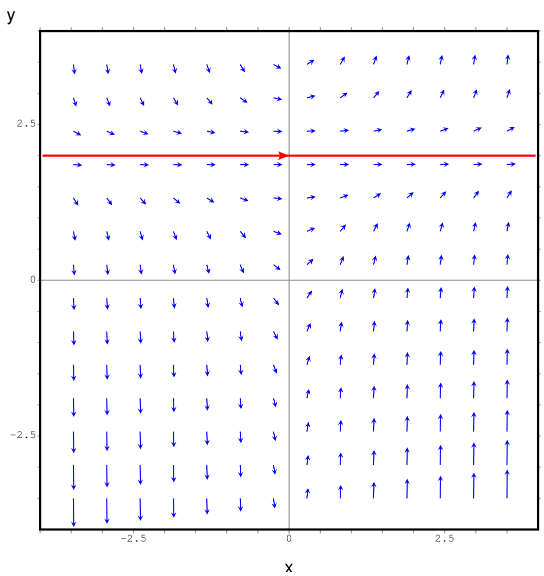
(d) 0

(e) 1

(f) $2 \longrightarrow$ **CORRECT**

(g) 3

Solution: The key point here is to not try to solve the equation. You can see that $y = 2$ is an equilibrium solution.



4. Consider the differential equation below

$$y' = \frac{(y-1)^{2/3}}{x}$$

Determine which of the initial values will guarantee the existence of a unique solution for x near that initial value.

(a) $y(0) = 0$
(b) $y(0) = 1$
(c) $y(1) = 0$ $\longrightarrow$ **CORRECT**
(d) $y(1) = 1$
(e) $y(2) = 1$
(f) $y(0) = 2$
(g) None of the above guarantee a unique solution

Solution: According to the theorem, there will exist a unique solution if $f(x, y)$ and $f_y(x, y)$ are both continuous on a rectangle containing the initial value. The only initial value that satisfies this is $(1, 0)$.

5. Consider the initial value problem

$$xy' - 2y = 0, \quad y(1) = 1$$

Which of the following statements are true?

I. The equation has no solution

II. The equation has a unique solution defined on $(0, 2)$

III. The equation has a unique solution defined on $\mathbb{R}$

(a) I only

(b) II only $\longrightarrow$ **CORRECT**

(c) III only

(d) I and II only

(e) I and III only

(f) II and III only

(g) All statements are true

(h) None of the statements are true

Solution: This is a first order linear equation: $y' - \frac{2}{x}y = 0$. According to the theorem on existence and uniqueness for first order linear equations, the equation $y' + P(x)y = Q(x)$ has a unique solution when P and Q are both continuous. Thus, there is a problem at $x = 0$. This is why answer II. is true—there is a unique solution on $(0, 2)$.

You can solve the equation and find the solution $y = Ax^2$. But, the key point to the answer is that you can make any of these parabolas match up at the origin and thus get infinitely many solutions defined on the real line.

6. Which of the following differential equations is separable?

   (a) $y' - e^y = e^{x+y}$ $\longrightarrow$ **CORRECT**

   (b) $x\dfrac{dy}{dx} - 2y = x + 2y$

   (c) $xy\dfrac{dy}{dx} = x^3 + y^3$

   (d) $(x + y)y' = x - y$

   (e) $y' = \ln(xy)$

   (f) None of the above are separable

7. Which of the equations below is exact?

   I. $-\dfrac{y}{x^2}\,dx + \dfrac{1}{x}\,dy = 0$

   II. $(2xy - e^x) + x^2\,\frac{dy}{dx} = 1$

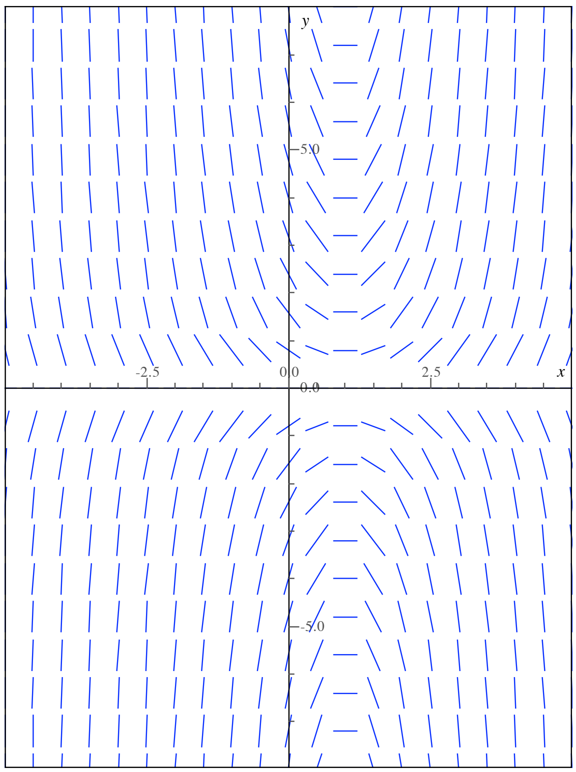
   III. $x\,dx - y\,dy = 0$

   IV. $y\,dx - x\,dy = 0$

   (a) I only
   (b) II only
   (c) III only
   (d) IV only
   (e) I and II only
   (f) I and III only
   (g) I and IV only
   (h) II and III only
   (i) II and IV only
   (j) III and IV only
   (k) Some other combination or none are exact $\longrightarrow$ **CORRECT I, II, III**

   Solution: Remember an equation $M\,dx + N\,dy = 0$ is exact if $M_y = N_x$. So, just check to see which equations satisfy this condition.

8. Pick the differential equation that corresponds to the given slope field.



(a) $y' = yx$

(b) $y' = y(x+1)$

(c) $y' = y(x-1)$ $\longrightarrow$ **CORRECT**

(d) $y' = (y+1)x$

(e) $y' = (y+1)(x+1)$

(f) $y' = (y+1)(x-1)$

(g) $y' = (y-1)x$

(h) $y' = (y-1)(x+1)$

Solution: The key points to notice are where $y' = 0$, $y' > 0$ and $y' < 0$.

9. Torricelli's Law can be formulated as a statement about the rate that water drops in a water tank as the water drains from a hole in the tank. This formulation of Torricelli's law is

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}$$

where y is the height of the water, $A(y)$ is the cross-sectional area of the tank at height y, $g = 32$ ft/s^2 (acceleration due to gravity), and a is the area of the hole through which the water is draining.

An upright cylindrical water tank with a radius of 1 ft and a height of 4 ft and at time $t = 0$ is full of water. At that moment a circular hole with radius 1 inch (1/12 ft) is punched in the bottom of the tank. How long will it take for all of the water to drain from the tank?

(a) $t = 8$ s
(b) $t = 12$ s
(c) $t = 24$ s
(d) $t = 32$ s
(e) $t = 64$ s
(f) $t = 72$ s $\longrightarrow$ **CORRECT**

Solution: For this tank, we have $A(y) = \pi$ and $a = \pi/(12^2)$. The differential equation becomes

$$\pi\frac{dy}{dt} = -\frac{\pi}{144}\sqrt{64y}, \qquad y(0) = 4$$

Solving give a general solution of $y = (C - t/36)^2$. The initial condition gives particular solution $y = (2 - t/36)^2$. Setting $y = 0$ and solving tells us that the tank is empty when $t = 72$.

10. Solve the initial value problem

$$\frac{dy}{dx} - \frac{2}{x}y = x^2e^x, \qquad y(1) = e$$

What is $y(2)$? Round your answer to one decimal place.

(a) 2.7
(b) 5.1
(c) 20
(d) 21.7
(e) 25.1
(f) 29.6 $\longrightarrow$ **CORRECT**

Solution: First order linear

$$\begin{aligned} y &= x^2(e^x + C) \\ y &= x^2e^x \\ y(2) &= 2^2e^2 \approx 29.556224 \end{aligned}$$

11. Solve the initial value problem

$$\frac{dy}{dx} = x\sqrt{x^2+9}, \qquad y(-4) = 0.$$

Find $y(0)$. Round your answer to one decimal place.

(a) -33.7
(b) -32.7 $\longrightarrow$ **CORRECT**
(c) -31.7
(d) 31.7
(e) 32.7
(f) 33.7

Solution: Separation of variables (or just integrate the equation).

$$y = \int x\sqrt{x^2+9}\,dx = \tfrac{1}{3}(x^2+9)^{3/2} + C$$
$$y = \tfrac{1}{3}(x^2+9)^{3/2} - \tfrac{125}{3}$$
$$y(0) = -\tfrac{98}{3} \approx -32.6667$$

12. A 200L tank contains 10 kg of salt dissolved in 100L of water. Pure water is pumped into the tank at the rate of 3 L/s, and the uniform mixture is pumped out at a rate of 2 L/s. How much salt is in the tank when it is full? Pick the closest answer.

(a) 0 kg
(b) 1 kg
(c) 1.5 kg
(d) 2 kg
(e) 2.5 kg $\longrightarrow$ **CORRECT**
(f) 3 kg
(g) 3.5 kg
(h) 4 kg

Solution: The differential equation modeling this is

$$\frac{dx}{dt} = \frac{2x}{100+t}, \qquad x(0) = 10$$

Solving gives $x = C(t+100)^{-2}$ with particular solution $x = 10^5(100+t)^{-2}$. Plugging in $t = 100$, $x(100) = 2.5$.

13. Consider the equation

$$2x - 2yy' = 0$$

Exactly one solution curve (trajectory) passes through the point $(2,1)$. Which of the following points is also on that solution curve (trajectory)?

(a) $(-1,-2)$
(b) $(-1,-1)$
(c) $(0,0)$
(d) $(1,1)$
(e) $(1,2)$
(f) $(\sqrt{2},\sqrt{5})$
(g) $(\sqrt{5},\sqrt{2})$ $\longrightarrow$ **CORRECT**
(h) None of these line on the same solution curve as the point $(2,1)$

Solution: This is equation is either separable or exact. In either case, the general solution is $x^2 - y^2 = C$. The particular solution we are looking for is $x^2 - y^2 = 3$. Thus we see that only $(\sqrt{5},\sqrt{2})$ is on this solution curve.

14. Find the solution to

$$y' = \frac{xy - y^2}{x^2}, \quad y(1) = \frac{1}{2}$$

Find $y(2)$ and round your answer to two decimal places.

(a) 0
(b) 0.21
(c) 0.53
(d) 0.67
(e) 0.70
(f) 0.74 $\longrightarrow$ **CORRECT**
(g) 0.83
(h) 0.94
(i) 1.0

Solution: Homogeneous (or Bernoulli). Substitution of $v = y/x$ yields

$$\begin{aligned} v + x\tfrac{dv}{dx} &= v - v^2 \\ x\tfrac{dv}{dx} &= -v^2 \\ \tfrac{1}{v} &= \ln x + C \\ \tfrac{x}{y} &= \ln x + C \\ \tfrac{x}{y} &= \ln x + 2 \\ y &= \tfrac{x}{\ln x + 2} \\ y(2) &= \tfrac{2}{\ln 2 + 2} \approx 0.7426256 \end{aligned}$$

WRITTEN PROBLEM—SHOW YOUR WORK

Name:
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Note: You will be graded on the readability of your work. Use the back of this sheet, if necessary.

15. Consider the autonomous equation $x'(t) = \dfrac{2x^4 - 3x^3 - 20x^2}{50}$

   (a) Find all the equilibrium solutions.

   (b) Sketch the slope field corresponding to this equation. Include several possible solution curves.

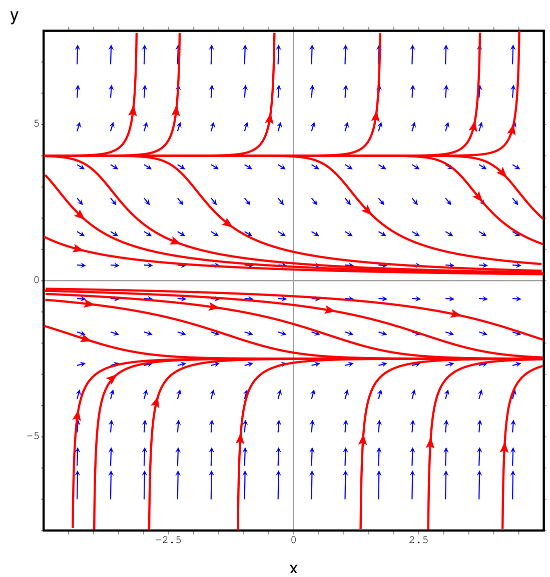
   (c) For each solution that you found in part (a), state whether it is stable, unstable, or semistable.

Solution: Factor the equation

$$x' = \frac{x^2(x-4)(2x+5)}{50}$$

Thus you can see the equilibrium solutions are $x = 0$, $x = 4$ and $x = -5/2$.

$x = 0$ is semi-stable. $x = 4$ is unstable. $x = -5/2$ is stable.



WRITTEN PROBLEM—SHOW YOUR WORK

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Note: You will be graded on the readability of your work. Use the back of this sheet, if necessary.

16. A tank initially contains 60 gal of pure water. Brine containing 1 lb/gal of salt enters the tank at 1 gal/min, and the well mixed solution leaves the tank at 2 gal/min.

    (a) How long does it take for the tank to empty?

    (b) Find the amount of salt, $x(t)$ in the tank after t minutes.

    (c) What is the maximum amount of salt ever in the tank?

Solution: $V(t) = 60 - t$ and the tank empties in 60 minutes.

The differential equation is

$$x' = 1 - \frac{2x}{60-t}, \quad x(0) = 0$$

Solving gives general and particular solutions of

$$\begin{aligned} x &= \quad (60-t)^2 \left[C + \tfrac{1}{60-t}\right] \\ x &= (60-t)^2 \left[-\tfrac{1}{60} + \tfrac{1}{60-t}\right] \end{aligned}$$

To find the maximum, set the derivative to 0:

$$x' = 1 - 2(60-t)\left(-\frac{1}{60} + \frac{1}{60-t}\right) = 0$$

Solving gives $t = 30$. The amount of salt in the tank at that time is

$$x(30) = 15$$