Math 217 - December 9, 2010

1. Let $y = \sum c_n x^n$ be nonzero series solution to $(x-3)y' + 2y = 0$. What is the radius of convergence of y?
2. Find the recursion relation for a series solution centered at $x = 0$.

$$(x-1)y'' + (x-2)y' + y = 0$$

1. Let $y = \sum c_n x^n$ be nonzero series solution to $(x-3)y' + 2y = 0$. What is the radius of convergence of y?
   Solution: Use the theorems, 3.
2. Find the recursion relation for a series solution centered at $x = 0$.

$$(x-1)y'' + (x-2)y' + y = 0$$

   Solution:

$$c_{n+2} = \frac{(n-2)c_{n+1} + c_n}{n+2}$$

3. For each of the equations below determine all singular points (all other points are ordinary). For each singular point, $x = a$ do the following:
   - For each singular point, translate the equation using $t = x - a$ and put the equation in the form $t^2 y'' + tp(t)y' + q(t)y = 0$.
   - Determine $p(t)$ and $q(t)$. Determine if $x = a$ is a regular singular point or an irregular singular point.
   - If $x = a$ is a regular singular point, find p_0 and q_0. Determine the exponents of the singular point.

   (a) $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0.$
   (b) $xy'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0.$
   (c) $(x^2 - 1)y'' + y' - y = 0.$
   (d) $(x^2 - 1)y'' + \frac{1}{x-1}y' - y = 0.$
   (e) $(x^2 - 1)y'' + y' - \frac{1}{x+1}y = 0.$
   (f) $y'' + \frac{1}{\sin \pi x}y' - y = 0.$
   (g) $x^2(x^2 - 1)y'' + \frac{1}{\sin \pi x}y' - 2y = 0.$

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- If $x = a$ is a regular singular point, find p_0 and q_0. Determine the exponents of the singular point.

(a) $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$.
Solution: Singular points: $x = 0$, exponents: $-1, 1$

(b) $xy'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$.
Solution: Singular points: $x = 0$, irregular.

(c) $(x^2 - 1)y'' + y' - y = 0$.
Solution: Singular points: $x = 1$ with exponents $0, 1/2$, $x = -1$ with exponents $0, 3/2$.

(d) $(x^2 - 1)y'' + \frac{1}{x-1}y' - y = 0$.
Solution: Singular points: $x = 1$ (irregular), $x = -1$ with exponents $0, 3/4$.

(e) $(x^2 - 1)y'' + y' - \frac{1}{x+1}y = 0$.
Solution: Singular points: $x = 1$ with exponents $0, 1/2$, $x = -1$ with exponents $1, 1/2$.

4. x is population with independent variable t, time. Discuss stability and population of the models below.

   (a) $x' = x^2 - 2x - 1$
   (b) $x' = x^2 - 2x + 1$
   (c) $x' = x^2 - 2x + 2$
   (d) $x' = x^3 - 2x^2 + x$
   (e) $x' = (x-4)^2(x-5)(6-x)$

5. A 200L tank contains a 5 g/L salt solution. A 20 g/L salt solution is pumped in at 10 L/min and the tanks is drained at 10 L/min. Find an expression for the smount of salt in the tank as a function of time.
6. Find a power series solution to $xy'' + 3xy' - y = 0$ centered at $x = 1$. (Just worry about finding the recursion relation and the first several terms).

5. A 200L tank contains a 5 g/L salt solution. A 20 g/L salt solution is pumped in at 10 L/min and the tanks is drained at 10 L/min. Find an expression for the smount of salt in the tank as a function of time.
   Solution: $x = 4000 - 3000e^{-t/20}$
6. Find a power series solution to $xy'' + 3xy' - y = 0$ centered at $x = 1$. (Just worry about finding the recursion relation and the first several terms).
   Solution:
   $a_{n+2} = -(n+3)(n+2)a_{n+1} - (3n-1)/((n+2)(n+1))a_n$

7. Compute the convolution $(\sin 5t) * (\cos 5t)$.
8. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

9. Convert the equation, using the Laplace transform, to a differential equation in $\mathscr{L}\{x\} = X$.

$$tx'' + x' + tx = 0, \qquad x(0) = 1, x'(0) = 0$$

7. Compute the convolution $(\sin 5t) * (\cos 5t)$.
   Solution: $\frac{1}{2} t \sin 5t$
8. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

   Solution: $x = \begin{bmatrix} 12 \\ -6 \end{bmatrix} e^{2t} + \begin{bmatrix} -8 \\ 8 \end{bmatrix} e^{3t}$
9. Convert the equation, using the Laplace transform, to a differential equation in $\mathscr{L}\{x\} = X$.

$$tx'' + x' + tx = 0, \qquad x(0) = 1, x'(0) = 0$$

   Solution: $(s^2 + 1)X' + sX = 0$

10. Solve $ye^{xy}\, dx + xe^{xy}\, dy = 0$
11. Find $\mathscr{L}^{-1}\{1/\sqrt{s-1}\} =$
12. Determine the value of m that will yield resonance

$$mx'' + x = 10\cos 3t$$

13. The equation

$$y'' - \frac{2}{x}y' + \frac{2}{x^2} = \sqrt{x}$$

and the complementary solution $y_c = c_1 x + c_2 x^2$, find a particular solution.

10. Solve $ye^{xy}\,dx + xe^{xy}\,dy = 0$
    Solution: Equation is exact
11. Find $\mathscr{L}^{-1}\{1/\sqrt{s-1}\} = e^t/\sqrt{\pi t}$
12. Determine the value of m that will yield resonance

$$mx'' + x = 10\cos 3t$$

    Solution: $m = \frac{1}{9}$
13. The equation

$$y'' - \frac{2}{x}y' + \frac{2}{x^2} = \sqrt{x}$$

    and the complementary solution $y_c = c_1 x + c_2 x^2$, find a particular solution.
    Solution: Variation of parameters, $y = \frac{4}{3}x^{5/2}$

14. Find a general solution to $y' = 1 + 2x + y + 2xy$. Hint: separable.

15. Solve $y'' + 2y' + y = 0$, $y(0) = 5, y'(0) = -3$.

14. Find a general solution to $y' = 1 + 2x + y + 2xy$. Hint: separable.
Solution: $\ln|1+y| = x + x^2 + C$
15. Solve $y'' + 2y' + y = 0$, $y(0) = 5, y'(0) = -3$.
Solution: $y = 5e^{-t} + 2te^{-t}$

16. Match the slope fields and the differential equations (the ranges for each graph are all $-2 \leq x, y \leq 2$).

$$y' = x+2y, \qquad y' = x-2y, \qquad y' = x+4y, \qquad y' = x-4y,$$

16. Match the slope fields and the differential equations (the ranges for each graph are all $-2 \leq x, y \leq 2$).

$$y' = x+2y, \qquad y' = x-2y, \qquad y' = x+4y, \qquad y' = x-4y,$$

Solution:

1 $\rightarrow$South West

2 $\rightarrow$North East

3 $\rightarrow$North West

4 $\rightarrow$South East