Math 217 - December 6, 2010

Theorem (Frobenius Series)

Suppose $x = 0$ is a regular singular point of the equation $x^2y'' + xp(x)y' + q(x)y = 0$. Let r_1 and r_2 be the roots, $r_1 \geq r_2$ of the indicial equation $r(r-1) + p_0 r + q_0 = 0$. Then

1. *There exists a solution to the differential equation of the form $y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^2$.*
2. *If $r_1 - r_2 \notin \mathbb{Z}$, there there exists a second linearly independent solution of the form $y_1 = x^{r_2} \sum_{n=0}^{\infty} b_n x^2$.*
3. *If $r_1 - r_2 \in \mathbb{Z}$ then a second linearly independent solution can be found using methods in section 8.4.*

1. What does the above theorem say about solutions to the differential equation

$$(\sin x)y'' + \frac{1}{x}y' + y = 0$$

Lecture Problems

2. Using the root $r = -1$, find the Frobenius series solution to

$$2x^2y'' + 3xy' - (x^2 + 1)y = 0$$

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Solution:

$$y = b_0x^{-1}\left(1 + \frac{1}{2}x^2 + \frac{1}{40}x^4 + \frac{1}{2160}x^6 + \frac{1}{224640}x^8 + \cdots\right)$$

3. Find a Frobenius solution to

$$xy'' + e^x y = 0$$

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Solution: Roots: $1, 0$.

$$y = a_0 x \left(1 - \frac{1}{2}x - \frac{1}{12}x^2 + \frac{1}{144}x^3 + \frac{23}{2880}x^4 + \frac{197}{86400}x^5 + \cdots\cdots \right)$$

4. Find a Frobenius solution to

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Solution: Roots: $1/2, 1/2$.

$$y = x^{1/2}$$

Side note, a second linearly independent solution is

$$y_2 = x^{1/2}\left(\ln x + \sum_{n=1}^{\infty} \frac{4^n x^n}{n!n}\right)$$