Math 132: Discussion Session: Week 13

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn this paper in at the end of the class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:  
<table>
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<th>Correct Answers</th>
<th>Grade</th>
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Group Members:

1a.) 1c.)
1b.) 1d.)
2.  
3a.) \[\sum_{n=0}^{\infty} 0n\] 3b.) \[\sum_{n=0}^{\infty} 4n\]
4a.) 4d.)
4b.) 4e.)
4c.)
5.)
6a.) 6c.)
6b.) 6d.)
7.)
8a.) 8b.) 8c.)
9a.) 9b.) 9c.)
10a.) 10b.) 10c.)
11a.) 11b.) 11c.)
12a.) 12b.) 12c.)
Math 132: Discussion Session: Week 13 Problem Sheet

Directions: In groups of 3-4 students, work the problems on this page.

1.) Write all letters that apply in each situation:
   A.) Divergence Test    E.) Comparison Test    H.) Alternating Series Test
   B.) Geometric Series   F.) Limit Comparison Test I.) Ratio Test
   C.) P-series           G.) Integral Test      J.) Root Test
   D.) Telescoping Series

   a.) You could find the actual value that the series converges to
   b.) Given that you are not absolutely convergent, use _____ to find out if your series is conditionally convergent
   c.) Typically use ___________ when you see $n!$ as part of the expression of $a_n$
   d.) Typically use ___________ when you see $n$ as part of an exponent in the expression of $a_n$

2.) Use the integral test to determine for which values of $p$ the series $\sum_{n=1}^{\infty} \frac{1}{n^{[\ln(9n)]^p}}$ will converge.

3.) Rewrite each alternating series in summation notation with an index starting at 0.
   a.) $\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \ldots$
   b.) $-\frac{\sqrt{1}}{1} + \frac{\sqrt{2}}{2} - \frac{\sqrt{4}}{3} + \frac{\sqrt{8}}{4} - \frac{\sqrt{16}}{5} + \ldots$

4.) Determine if the following series converge or diverge.
   a.) $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$
   d.) $\sum_{n=3}^{\infty} \frac{\ln n}{n}$
   b.) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$
   e.) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$
   c.) $\sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2}$

5.) Does convergence change for any of the series in question 4 if a factor of $(-1)^{n-1}$ is multiplied by the terms in the series? If so, for which ones? (List them by the letter.)
6.) Do the following series diverge, converge absolutely, or converge conditionally?
   a.) \( \sum_{n=1}^{\infty} \frac{-1(-3)^n}{n^n} \)
   b.) \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n + 2}{5n + n^2} \)
   c.) \( \sum_{n=2}^{\infty} (-2)^{n+1} \frac{\sin(n + 2)}{n^2} \)
   d.) \( \sum_{n=0}^{\infty} \text{sech } n \) (Hint: \( \text{sech } n = \frac{2}{e^n + e^{-n}} \))

7.) Solve and write the answer in interval notation: \( |2x - 7| < 1 \)

Consider the following series. Answer these three questions for each one…
   a.) If \( x = 10 \), does the series converge or diverge?
   b.) If \( x = 0.1 \), does the series converge or diverge?
   c.) For what values of \( x \) does the series converge? (Express in interval notation.)

8.) \( \sum_{n=0}^{\infty} x^n \)

9.) \( \sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n \)

10.) \( \sum_{n=0}^{\infty} \frac{x^n}{13^n+1} = \sum_{n=0}^{\infty} \frac{x^n}{13 \cdot 13^n} = \sum_{n=0}^{\infty} \frac{1}{13} \left( \frac{x}{13} \right)^n \)

11.) \( \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n} = \sum_{n=0}^{\infty} \left( \frac{x-1}{2} \right)^n \)

12.) \( \sum_{n=1}^{\infty} \frac{x^n}{n} \)
Math 132: Discussion Session: Week 13 Problem Sheet

Directions: In groups of 3-4 students, work the problems on this page.

1.) Write all letters that apply in each situation:
   A.) Divergence Test  E.) Comparison Test  H.) Alternating Series Test
   B.) Geometric Series  F.) Limit Comparison Test  I.) Ratio Test
   C.) P-series  G.) Integral Test  J.) Root Test
   D.) Telescoping Series

   a.) You could find the actual value that the series converges to
      B, D
   b.) Given that you are not absolutely convergent, use _____ to find out if your series is
      conditionally convergent
      H
   c.) Typically use ___________ when you see $n!$ as part of the expression of $a_n$
      I
   d.) Typically use ___________ when you see $n$ as part of an exponent in the expression of $a_n$
      B, I, J

2.) Use the integral test to determine for which values of $p$ the series $\sum_{n=1}^{\infty} \frac{1}{n[\ln(9n)]^p}$ will converge.

   \[
   \int_{1}^{\infty} \frac{1}{x[\ln(9x)]^p} \, dx = \int_{\ln 9}^{\infty} \frac{1}{u^p} \, du
   \]
   \[
   u = \ln(9x)
   \]
   \[
   du = \frac{9}{9x} \, dx = \frac{1}{x} \, dx
   \]

   For $p > 1$:
   \[
   = \left. \frac{1}{(p-1)u^{p-1}} \right|_{\ln 9}^{\infty} \quad \text{which converges.}
   \]

   For $p < 1$:
   \[
   = \left. \frac{u^{1-p}}{(1-p)\ln 9} \right|_{\ln 9}^{\infty} \quad \text{which diverges.}
   \]

   Therefore this series will converge for $p > 1$

3.) Rewrite each alternating series in summation notation with an index starting at 0.

   a.) \[
   \frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)!}
   \]

   b.) \[
   -\sqrt{\frac{1}{1}} + \sqrt{\frac{2}{2}} + \sqrt[3]{\frac{3}{4}} + \sqrt[4]{\frac{4}{5}} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2)^{n+2}}{n+1}
   \]
4.) Determine if the following series converge or diverge.

a.) \[ \sum_{n=2}^{\infty} \frac{1}{n^2 \ln n} \]
   Converge by direct comparison
to \[ \sum_{n=2}^{\infty} \frac{1}{n^2} \] when \( n \geq 3 \)

b.) \[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \]
   Diverge by the integral test

c.) \[ \sum_{n=2}^{\infty} \frac{1}{n \left( \ln n \right)^2} \]
   Converge by the integral test

d.) \[ \sum_{n=2}^{\infty} \frac{\ln n}{n} \]
   Diverge by direct comparison
to \[ \sum_{n=2}^{\infty} \frac{1}{n} \] when \( n \geq 3 \)

e.) \[ \sum_{n=2}^{\infty} \frac{n}{\ln n} \]
   Diverge by the divergence test

5.) Does convergence change for any of the series in question 4 if a factor of \((-1)^{n-1}\) is multiplied by the terms in the series? If so, for which ones? (List them by the letter.)

Yes, for (b) and (d), the Alternating Series Test would show convergence

For (b)… \[ \frac{1}{(n+1) \ln(n+1)} \leq \frac{1}{n \ln n} \] and \( \lim_{n \to \infty} \frac{1}{n \ln n} = 0 \)

For (d)… Consider the function \( f(x) = \frac{\ln x}{x} \). \( f'(x) = \frac{1 - \ln x}{x^2} < 0 \) for \( x \geq 3 \) and therefore decreasing. Also, \( \lim_{n \to \infty} \frac{\ln n}{n} = \lim \frac{1/n}{1} = 0 \)

6.) Do the following series diverge, converge absolutely, or converge conditionally?

a.) \[ \sum_{n=1}^{\infty} \frac{-1(-3)^n}{n^n} \]
   By the root test, \( \lim_{n \to \infty} \sqrt[n]{\frac{-1(-3)^n}{n^n}} = \lim_{n \to \infty} \frac{3}{n} = 0 < 1 \), converges absolutely

b.) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n+2}{5n+n^2} \]
   \( \sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{3n+2}{5n+n^2} \right| = \sum_{n=1}^{\infty} \frac{3n+2}{5n+n^2} \)
   Limit compare to \( \sum_{n=1}^{\infty} \frac{1}{n} \), a divergent \( p \)-series
   \( \lim_{n \to \infty} \frac{3n+2}{5n+n^2} \cdot \frac{n}{1} = 3 \). Therefore \( \sum_{n=1}^{\infty} \frac{3n+2}{5n+n^2} \) diverges.
   But, if we let \( f(x) = \frac{3x+2}{5x+x^2} \), then
\[
\frac{d}{dx} f(x) = \frac{3(5x + x^2) - (3x + 2)(5 + 2x)}{(5x + x^3)^2} = \frac{-(3x^2 + 4x + 10)}{(5x + x^3)^2}
\]
which is negative for \(x \geq 1\). Therefore, this function is decreasing, and so the series is decreasing.

Also, \(\lim_{n \to \infty} \frac{3n + 2}{5n + n^2} = 0\). So, by the alternating series test, \(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n + 2}{5n + n^2}\) converges. Therefore, the convergence is conditional.

c.) \(\sum_{n=2}^{\infty} (-2)^{n+1} \frac{\sin n + 2}{n^2}\)

\[
\sum_{n=2}^{\infty} (-2)^{n+1} \left( \frac{\sin n + 2}{n^2} \right) = \sum_{n=2}^{\infty} (-1)^{n+1} 2^{n+1} \left( \frac{\sin n + 2}{n^2} \right)
\]
Notice that \(\frac{2^{n+1}(1)}{n^2} \leq \frac{2^{n+1}(\sin n + 2)}{n^2} \leq \frac{2^{n+1}(3)}{n^2}\)

\[
\lim_{n \to \infty} \frac{2^{n+1}}{n^2} = \lim_{n \to \infty} \frac{2^{n+1} \ln(2)}{2n} = \lim_{n \to \infty} \frac{2^{n+1} (\ln 2)^2}{2} = \infty
\]
Therefore, \(\lim_{n \to \infty} \frac{2^{n+1} (\sin n + 2)}{n^2} \geq \lim_{n \to \infty} \frac{2^{n+1}}{n^2} = \infty\) and so this diverges by the alternating series test.

d.) \(\sum_{n=0}^{\infty} \text{sech } n\) (Hint: \(\text{sech } n = \frac{2}{e^n + e^{-n}}\))

Note that \(\sum_{n=0}^{\infty} |\text{sech } n| = \sum_{n=0}^{\infty} \frac{2}{e^n + e^{-n}} = \sum_{n=0}^{\infty} \frac{2e^n}{e^{2n} + 1}\)

Limit compare to \(\sum_{n=0}^{\infty} \frac{1}{e^n}\), a convergent geometric series

\[
\lim_{n \to \infty} \frac{2e^n}{e^{2n} + 1} = \lim_{n \to \infty} \frac{2e^n}{e^{2n} + 1} = \lim_{n \to \infty} \frac{4e^{2n}}{2e^{2n}} = 2.
\]
Therefore \(\sum_{n=0}^{\infty} \text{sech } n\) converges absolutely.

7.) Solve and write the answer in interval notation:
\(2x - 7 < 1\)
\(-1 < 2x - 7 < 1\)
\(6 < 2x < 8\)
\(3 < x < 4\) \(\Rightarrow (3, 4)\)
Consider the following series. Answer these three questions for each one…

a.) If \( x = 10 \), does the series converge or diverge?

b.) If \( x = 0.1 \), does the series converge or diverge?

c.) For what values of \( x \) does the series converge? (Express in interval notation.)

8.) \( \sum_{n=0}^{\infty} x^n \)

(a) diverge  
(b) converge  
(c) Using the geometric test, we see that it converges when \( |x| < 1 \Rightarrow (-1,1) \) (Note that if we tried ratio/root tests, we would get the same results, but we might also want to check to see what happens when the test fails (i.e., when \( |x| = 1 \)). In both cases, \( \sum_{n=0}^{\infty} (-1)^n \) and \( \sum_{n=0}^{\infty} (1)^n \) diverge. Neither endpoint would be included.

9.) \( \sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n \)

(a) diverge  
(b) converge  
(c) \( |4x| < 1 \Rightarrow \left( -\frac{1}{4}, \frac{1}{4} \right) \)

10.) \( \sum_{n=0}^{\infty} \frac{x^n}{13^n+1} = \sum_{n=0}^{\infty} \frac{x^n}{13 \cdot 13^n} = \sum_{n=0}^{\infty} \frac{1}{13} \left( \frac{x}{13} \right)^n \)

(a) converge  
(b) converge  
(c) \( \left| \frac{x}{13} \right| < 1 \Rightarrow (-13,13) \)

11.) \( \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n} = \sum_{n=0}^{\infty} \left( \frac{x-1}{2} \right)^n \)

(a) diverge  
(b) converge  
(c) \( \left| \frac{x-1}{2} \right| < 1 \Rightarrow -2 < x-1 < 2 \Rightarrow (-1,3) \)

12.) \( \sum_{n=1}^{\infty} \frac{x^n}{n} \)

(a) diverge  
(b) converge  
(c) This is tricky…It’s not geometric anymore. Notice that for \( |x| < 1 \) this will converge absolutely by the ratio/root test and for \( |x| > 1 \) it will diverge. What about when \( x = \pm 1 \) when the test fails? For \( x = 1 \), we have the harmonic series and therefore it diverges. But for \( x = -1 \), we have \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) which converges by the alternating series test. Therefore the interval is \([-1,1)\).