Section 23
11.4: Comparison Tests

1. **Solution:** Note that since $2n \leq n^2$ and $1 \leq n^2$ we have $n^2 + 2n + 1 \leq n^2 + n^2 + n^2 = 3n^2$.

2. **Solution:** A comparison test works fine since \[ \frac{n^2 \cdot 2n - 1}{n^4 + 2n + 1} \leq \frac{n^2}{n^4} = \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^2 - 2n - 1}{n^4 + 2n + 1} \]

3. **Solution:** The comparison test doesn’t work the following isn’t true: \[ \frac{n^2 + 2n + 1}{n^4 - 2n - 1} \leq \frac{n^2}{n^4} = \frac{1}{n^2} \]

   So, we can either do something to make the inequality correct or else do a limit comparison (more advanced, but easier). Here’s how we can make the inequality work.

   First, notice that for large values of $n$ we have the following inequalities:
   - $2n \leq n^2$ (true for $n \geq 2$)
   - $1 \leq n^2$ (true for $n \geq 1$)
   - $-2n \geq -\frac{1}{3}n^4$ (true for $n \geq 2$)
   - $-1 \geq -\frac{1}{3}n^4$ (true for $n \geq 2$)

   \[ \frac{n^2 + 2n + 1}{n^4 - 2n - 1} \leq \frac{n^2 + n^2 + n^2}{n^4 - \frac{1}{3}n^4 - \frac{1}{3}n^4} = \frac{3n^2}{\frac{1}{3}n^4} = \frac{9n^2}{n^4} \]

   and thus \[ \sum_{n=1}^{\infty} \frac{n^2 + 2n + 1}{n^4 - 2n - 1} \]
   converges because \[ \sum_{n=1}^{\infty} \frac{9n^2}{n^4} \]
   converges.

4. **Solution:** Well, you actually wouldn’t expect exact equality, but for large $n$ you would expect $a_n$ and $b_n$ to be very close: $a_n \approx b_n$.

5. **Solution:** If $L = 0$ then this means that, for large values of $n$, the denominator is much larger than the numerator: $a_n < b_n$.

6. **Solution:** If $L = \infty$ then this means that, for large values of $n$, the numerator is much larger than the denominator. $b_n < a_n$.

7.
Solution: \[ \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\left( \frac{n^2 - 2n - 1}{n^4 + 2n + 1} \right)}{\left( \frac{1}{n^2} \right)} = 1 \] which means that series behave the same and thus both converge.

8. 

Solution: \[ \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\left( \frac{n^2 + 2n + 1}{n^4 + 2n - 1} \right)}{\left( \frac{1}{n^2} \right)} = 1 \] which means that series behave the same and thus both converge.

9. (a) 

Solution: Diverges with a limit comparison to \( \frac{1}{n} \). You can also do a “normal” comparison to \( \frac{n^2}{2n} \) but that is much trickier.

(b) 

Solution: Diverges with a limit comparison to \( \frac{2n}{n^2} = \frac{2}{n} \).

(c) 

Solution: Converges with a limit comparison to \( \frac{1}{n^2} \).

(d) 

Solution: Converges with a limit comparison to \( b_n = \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{3/4}} \). Note, I choose \( n^{1/4} \) by some trial and error. Note that \( \lim_{n \to \infty} a_n = 0 \) and \( \sum b_n \) converges, so \( \sum a_n \) must converge too.