11.2: Series

1. **Solution:** First note that \( s_n \) is monotonic increasing. If it is also bounded then it converges. The following picture of a 1 \( \times \) 1 square should convince you that the sequence \( \{ s_n \} \) is bounded. And, it should also convince you that the sum is equal to 1.

\[
\begin{array}{c|c|c|c}
\frac{1}{64} & \frac{1}{16} & \frac{1}{4} \\
\frac{1}{32} & \frac{1}{8} & \\
\frac{1}{16} & & \\
\frac{1}{8} & & \\
\frac{1}{4} & & \\
\frac{1}{2} & & \\
\end{array}
\]

2. **Solution:**

\[
(1 - r)(1 + r + r^2 + \cdots + r^{n-1}) = 1 - r^n
\]

3. **Solution:** Long division here is a bit obnoxious but it is possible. It is easier to use the result from the previous problem.

\[
\frac{1 - r^n}{1 - r} = 1 + r + r^2 + \cdots + r^{n-1}
\]

4. **Solution:**

\[
\sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n = \frac{(2/3)}{1-(2/3)} = 2
\]

5. (a) **Answer:** \( \frac{4}{3} \)

(b) **Answer:** Diverges!

(c) **Answer:** 3

(d) **Answer:** \( \frac{1}{1+2/3} = \frac{3}{5} \)
(e) Answer: \[ \sum_{n=2}^{\infty} \left( \frac{3^n - 3}{5^n} \right) \left( \frac{2 \cdot 3}{25} \right)^n = \left( \frac{1}{9 \cdot 125} \right) \left( \frac{6}{25} \right)^2 \left( \frac{1}{1 - 6/25} \right) \]

6. (a) Answer: \[ \frac{(2/3) \cdot (1 - (2/3)^3)}{1 - (2/3)} \]
(b) Answer: \[ \frac{(1) \cdot (1 - (2/3)^3)}{1 - (2/3)} \]
(c) Answer: \[ \frac{(2/3)^3 \cdot (1 - (2/3)^3)}{1 - (2/3)} \]

7. Solution: This is geometric with \( a = x \) and \( r = x \) thus the sum is \( x/(1 - x) \).

Note that \( \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \). This series will be important.

8. Solution: Start with this part: \[ \frac{41}{10^3} + \frac{41}{10^7} + \frac{41}{10^{11}} + \cdots \]

For this portion, it is a geometric series with \( a = \frac{41}{10^3} \) and \( r = \frac{1}{10^4} \). Thus the sum of this portion is \( \frac{41/10^3}{1 - 1/10^4} = \frac{41}{999} \).

Thus the sum of the series is

\[
x = 3.2\overline{41} = 3.2 + 0.0\overline{41}
\]
\[
= \frac{32}{10} + \frac{41}{999} = \frac{3209}{999}
\]

Of course, you may have learned to do this differently, but the above is justification for the repeated decimal. Here is how I was taught to convert these repeating decimals to fractions when I was in middle school:

\[
100x = 324.1414141414141\ldots = 324.1\overline{41}
\]
\[
x = 3.2414141414141\ldots = 3.2\overline{41}
\]
\[
99x = 100x - x = 320.9
\]
\[
x = \frac{320.9}{99} = \frac{3209}{990}
\]

9. (a) Solution: Converges. This is geometric with \( r = \frac{99998}{99999} < 1 \).

(b) Solution: Note that

\[
\sum_{n=1}^{\infty} \left( \frac{3^n + 7^{n+1} - 23^{n+4}}{5^{2n+1}} \right) = \frac{1}{5} \sum_{n=1}^{\infty} \left[ \frac{1}{5} \left( \frac{3}{25} \right)^n + \frac{7}{5} \left( \frac{7}{25} \right)^n - \frac{23^4}{5} \left( \frac{23}{25} \right)^n \right]
\]

so this is a sum of geometric series, all of which converge.

(c)
10. (a) Answer: \( \frac{9}{10} \)
(b) Answer: 1

11. (a) Solution: \( s_N = 1 - \frac{1}{N} \)
\[ \sum_{n=1}^{\infty} \frac{1}{n^2 - n} = 1 \]
(b) Solution: \( s_N = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right) \)
\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 3} = \frac{5}{12} \]
(c) Solution: \( s_N = \ln n - \ln 1 \)
\[ \sum_{n=1}^{\infty} \ln \left( \frac{k + 1}{k} \right) = \text{Diverges} \]
This example is important since it illustrates that things can cancel out but the series diverges.