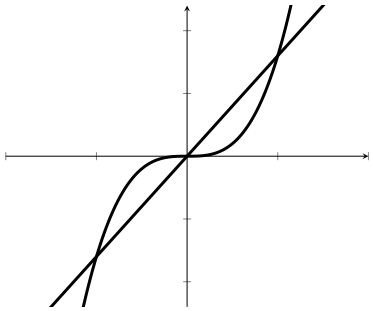
- 14 multiple choice questions worth 5 points each.
- 2 hand graded questions worth 15 points each.
- Exam covers sections 6.1, 6.2, 6.3, 6.5, 7.1, 7.2

- No calculators!
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. Your ability to make your solution clear will be part of your grade.

Useful Formulas

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$	$\sin^2\theta + \cos^2\theta = 1$
$1 + \tan^2\theta = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$
$\sin(A \pm B) = \sin A\cos B \pm \sin B\cos A$	$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$	$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$
$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$	$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$
$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
$\sin(2\theta) = 2\sin\theta\cos\theta$	$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\int \csc x\, dx = -\ln\lvert\csc x + \cot x\rvert + C$	$\int \sec x\, dx = \ln\lvert\sec x + \tan x\rvert + C$
$\cosh t = \frac{1}{2}(e^t + e^{-t})$	$\sinh t = \frac{1}{2}(e^t - e^{-t})$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$	$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$
$\cosh^2 t = 1 + \sinh^2 t$	

1. Find the area of the region enclosed by curve $y = 4x$ and curve $y = x^3$.



A. -10

B. -8

C. -4

D. 2

E. 0

F. 2

G. 4

H. 8

I. 10

Solution: intersections of two curves are $(-2, -8)$, $(0, 0)$, and $(2, 8)$

$$
\begin{aligned}
A &= \int_{-2}^{0} (x^3 - 4x)dx + \int_{0}^{2} (4x - x^3) \\
&= (\frac{1}{4}x^4 - 2x^2)\Big|_{-2}^{0} + (2x^2 - \frac{1}{4}x^4)\Big|_{0}^{2} \\
&= (0 - 4 + 8) + (8 - 4 - 0) \\
&= 4 + 4 \\
&= 8
\end{aligned}
$$

2. Find the area bounded by the curves $y = 3 - |x|$ and $y = x^2 + 1$.

   A. $\pi/2$

   B. 6

   C. 13/2

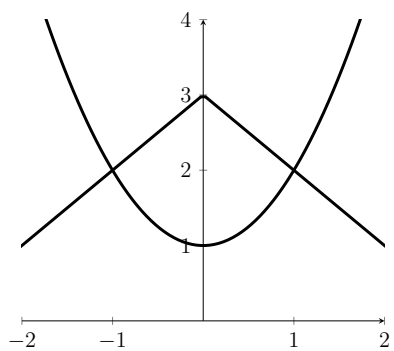
   D. 7/3

   E. 5

   F. 22/3

   G. $\pi^2/6$

Solution:

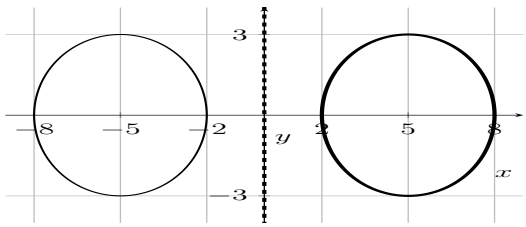


The curves intersect at $(1, 2)$ and $(-1, 2)$. Thus

$$\begin{aligned}
V &= \int_{-1}^{1} (3 - |x| - (x^2 + 1)\, dx \\
&= \int_{-1}^{1} -|x|\, dx + \int_{-1}^{1} (2 - x^2)\, dx \\
&= \int_{-1}^{0} -|x|\, dx + \int_{0}^{1} -|x|\, dx + \int_{-1}^{1} (2 - x^2)\, dx \\
&= \int_{-1}^{0} (x)\, dx + \int_{0}^{1} (-x)\, dx + \int_{-1}^{1} (2 - x^2)\, dx \\
&= -\frac{1}{2} - \frac{1}{2} + \frac{10}{3} = \frac{7}{3}
\end{aligned}$$

3. Cezareo needs to compute the volume of a donut. The donuts have a radius of 8 centimeters and when cut in half, there are two circular cross sections, each with radius of 3 centimeters.

   Which of the following integrals would help Cezareo determine the volume of a donuts using cylindrical shells?



   A. $2\pi \int_2^8 x\sqrt{8-(x-3)^2}\,dx$

   B. $4\pi \int_2^8 x\sqrt{8-(x-3)^2}\,dx$

   C. $2\pi \int_2^8 x\sqrt{9-(x-5)^2}\,dx$

   D. $4\pi \int_2^8 x\sqrt{9-(x-5)^2}\,dx$

   E. $\int_{-3}^3 \left[(x-3)^2-(3-x)^2\right]\,dx$

   F. $10\int_{-3}^3 \pi\sqrt{9-x^2}\,dx$

Solution: The equation of the circle in the right half plane is $(x-5)^2+y^2=3^2$. Solving for y gives

$$y=\pm\sqrt{9-(x-5)^2}$$

Where $y=\sqrt{9-(x-5)^2}$ is the upper part of the circle and $y=-\sqrt{9-(x-5)^2}$ is the lower half of the circle.

Setting up shells:

$$\begin{aligned} V &= \int 2\pi r h\,dr \\ &= \int_2^8 2\pi(x)\left[\sqrt{9-(x-5)^2}-(-\sqrt{9-(x-5)^2})\right]\,dx \\ &= 4\pi\int_2^8 x\sqrt{9-(x-5)^2}\,dx \end{aligned}$$

If you did this with washers, your integral would be

$$V = \pi \int_{-3}^{3} \left[(5 + \sqrt{9 - y^2})^2 - (5 - \sqrt{9 - y^2})^2 \right] \, dy$$

$$V = 20\pi \int_{-3}^{3} \sqrt{9 - y^2} \, dy$$

4. A solid is obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$ about the y-axis. Using **WASHERS/DISCS**, set-up the integral to calculate the volume of the solid.

A. $\pi \int_0^4 (\sqrt{x})^2 \, dx$

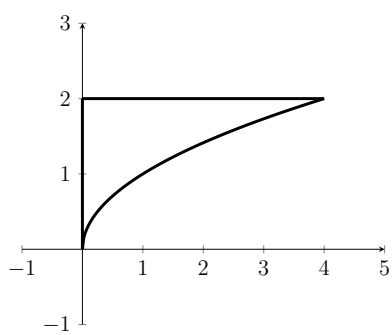
B. $\pi \int_0^2 y^4 \, dy$

C. $\pi \int_0^4 y^4 \, dx$

D. $\pi \int_0^2 y^2 \, dy$

E. $\pi \int_0^4 \sqrt{x} \, dx$

F. $\pi \int_0^2 (16 - y^4) \, dy$



Solution:

For washers:

$$V = \int_a^b \pi r^2 \, dy = \int_0^2 \pi (y^2)^2 \, dy$$

Using shells:

$$V = \int_a^b 2\pi r h \, dr = 2\pi \int_0^4 x(2 - \sqrt{x}) \, dx$$

5. A solid is obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$ about the y-axis. Using **CYLINDRICAL SHELLS**, set-up the integral to calculate the volume of the solid.

A. $2\pi \int_0^2 x(2 - \sqrt{x})\, dx$

B. $\pi \int_0^2 x(2 - \sqrt{x})\, dx$

C. $2\pi \int_0^4 x(2 - \sqrt{x})\, dx$

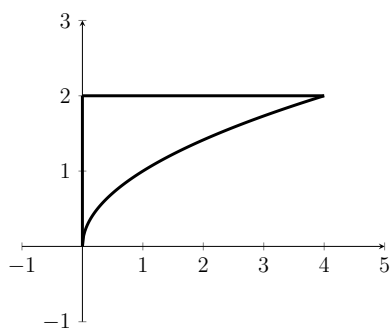
D. $\pi \int_0^4 x(2 - \sqrt{x})\, dx$

E. $2\pi \int_0^4 x\sqrt{x}\, dx$

F. $\pi \int_0^4 x\sqrt{x}\, dx$

G. $2\pi \int_0^2 y(2 - y^2)\, dy$

H. $\pi \int_0^2 y(2 - y^2)\, dy$



Solution:

For washers:

$$V = \int_a^b \pi r^2\, dy = \int_0^2 \pi (y^2)^2\, dy$$

Using shells:

$$V = \int_a^b 2\pi r h\, dr = 2\pi \int_0^4 x(2 - \sqrt{x})\, dx$$

6. Let R be the region enclosed by $y = \sin x$, $y = 0$, between $x = 0$ and $x = \pi$. Find an integral representing the volume of the solid formed by revolving the region about the axis $x = -\pi$.

A. $2\pi \int_0^{\pi} x \sin(x)\, dx$

B. $2\pi \int_0^{\pi} (x + \pi) \sin(x)\, dx$

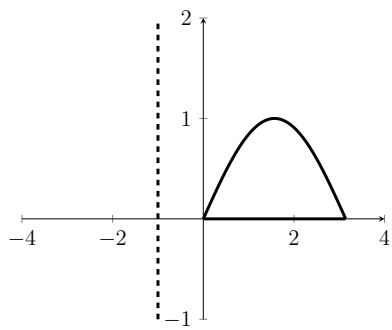
C. $2\pi \int_0^{\pi} (x - \pi) \sin(x)\, dx$

D. $2\pi \int_{-\pi}^{\pi} x \sin(x)\, dx$

E. $2\pi \int_{-\pi}^{\pi} (x + \pi) \sin(x)\, dx$

F. $2\pi \int_{-\pi}^{\pi} (x - \pi) \sin(x)\, dx$

G. $\pi \int_0^{1} (\sin^{-1} y)^2 - (\sin^{-1} y)^2\, dy$



Solution:

Using the shell method:

$$V(x) = \int 2\pi r h\, dr$$
$$= 2\pi \int_0^{\pi} x \sin(x)\, dx$$

7. Find the average value of $f(t) = t\cos(t^2)$ on the interval $[0, 10]$.

A. $\frac{1}{5}\cos(100)$

B. $\frac{1}{10}\cos(100)$

C. $\frac{1}{20}\cos(100)$

D. $\frac{1}{5}\sin(100)$

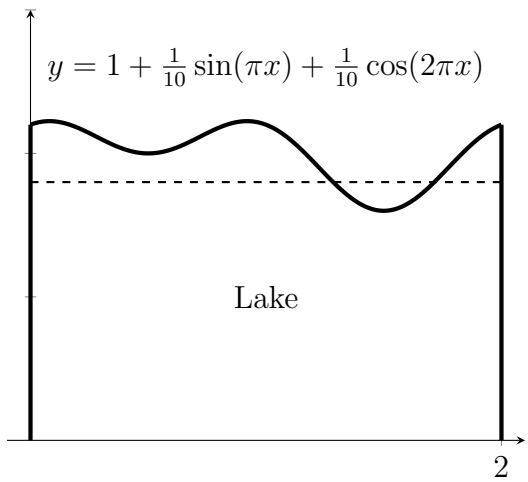
E. $\frac{1}{10}\sin(100)$

F. $\frac{1}{20}\sin(100)$

Solution:

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{10}\int_0^{10} t\cos(t^2)\,dt \\ &= \frac{1}{10}\left[\frac{1}{2}\sin(t^2)\right]_0^{10} \\ &= \frac{1}{20}\sin(100) \end{aligned}$$

8. Consider a small lake with waves. The lake is located on interval $[0, 2]$ on x-axis, and at a particular moment its water surface is given by equation $y = 1 + \frac{1}{10}\sin(\pi x) + \frac{1}{10}\cos(2\pi x)$. Find the equation of the surface when there are no waves and the water is perfectly calm (i.e., water surface is a straight line).



A. $y = 1$

B. $y = 1 + \dfrac{1}{10}\pi$

C. $y = 1 - \dfrac{1}{10}\pi$

D. $y = 1 + \dfrac{1}{5}\pi$

E. $y = 1 - \dfrac{1}{5}\pi$

Solution: Flat, the amount of water must be equal in both cases (the wave case and the flat water case). Thus, we are looking for the average height of the water, which we compute:

$$\begin{aligned} y_{\text{ave}} &= \frac{1}{2-0}\int_0^2 \left[1 + \frac{1}{10}\sin(\pi x) + \frac{1}{10}\cos(2\pi x)\right] dx \\ &= \frac{1}{2}\left[x - \frac{1}{10\pi}\cos(\pi x) + \frac{1}{20\pi}\sin(2\pi x)\right]\Bigg|_0^2 \\ &= \frac{1}{2}\left[(2 - \frac{1}{10\pi} + 0) - (0 - \frac{1}{10\pi} + 0)\right] \\ &= 1 \end{aligned}$$

9. Compute the definite integral

$$\int_1^e x^2 \ln(x)\, dx$$

A. e

B. $1 + 2e^2$

C. $1 - 2e^3$

D. $(1 + 2e^3)/3$

E. $(1 + 2e^3)/9$

F. $(1 + 2e^3)/12$

G. $\pi/3$

Solution: Use integration by parts with $u = \ln x$ and $dv = x^2\, dx$.

$$\begin{aligned}
\int_1^e x^2 \ln(x)\, dx &= \left[\frac{1}{3}x^3 \ln x\right]_1^e - \int_1^e \frac{1}{3}x^3 \cdot \frac{1}{x}\, dx \\
&= \left[\frac{1}{3}x^3 \ln x\right]_1^e - \int_1^e \frac{1}{3}x^2\, dx \\
&= \left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right]_1^e \\
&= \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}\right] = \left[\frac{1}{3}e^3 - \frac{1}{9}e^3\right] - \left[0 - \frac{1}{9}\right] = \frac{2}{9}e^3 + \frac{1}{9}
\end{aligned}$$

10. Compute the integral.

$$\int_0^{\ln 2} xe^x \, dx$$

A. $\ln 2$

B. $\ln 2 - 1$

C. $\ln 2 + 1$

D. $2\ln 2 - 1$

E. $2\ln 2 + 1$

Solution: Integration by parts with $u = x$ and $dv = e^x \, dx$.

$$\begin{aligned}\int_0^{\ln 2} xe^x \, dx =& xe^x - \int e^x \, dx \\ =& xe^x - e^x \Big|_0^{\ln 2} \\ =& (x-1)e^x \Big|_0^{\ln 2} = (\ln 2 - 1)(2) - (-1) = 2\ln 2 - 1\end{aligned}$$

11. Evaluate.

$$\int_0^{\pi} x^2 \cos(x)\, dx$$

A. π^2

B. $\pi^2 + 2\pi$

C. $2\pi - 2$

D. $\pi^2 - 2\pi - 2$

E. -2π

Solution:

$$\begin{aligned}
\int_0^{\pi} x^2 \cos x dx &= x^2 \sin x - 2 \int x \sin x dx \\
&= x^2 \sin x - 2 \left[-x \cos x + \int \cos x dx \right] \\
&= x^2 \sin x + 2x \cos x - 2 \sin x \Big|_0^{\pi} \\
&= (0 - 2\pi - 0) - (0 + 0 - 0) \\
&= -2\pi
\end{aligned}$$

12. Compute the integral.

$$\int_0^{\pi/4} \tan^2(x)\, dx$$

A. $\pi/4$

B. $1 + \pi/4$

C. $1 - \pi/4$

D. $\pi/2$

E. $1 + \pi/2$

F. $1 - \pi/2$

Solution:

$$\int_0^{\pi/4} \tan^2(x)\, dx = \int_0^{\pi/4} (\sec^2 x - 1)\, dx = \tan x - x \Big|_0^{\pi/4}$$
$$= (1 - \pi/4) - 0 = 1 - \pi/4$$

13. Compute the integral.

$$\int_{\pi/2}^{\pi} \sin x \cos^3 x \, dx$$

A. -1

B. $-1/2$

C. $-1/4$

D. $-1/8$

E. 0

F. $1/8$

G. $1/4$

H. $1/2$

I. 1

Solution: Use substitution $u = \cos x$.

$$\begin{aligned}\int_{\pi/2}^{\pi} \sin x \cos^3 x \, dx &= -\int u^3 \, du \\ &= -\frac{1}{4}u^4\Big|_{x=\pi/2}^{x=\pi} = -\frac{1}{4}\cos^4 x\Big|_{x=\pi/2}^{x=\pi} \\ &= -\frac{1}{4}(1) - 0 = -\frac{1}{4}\end{aligned}$$

14. Exactly one of the following integrals is not equal to zero. Which one?

A. $\displaystyle\int_{-3}^{3} x^5 \cos x \, dx.$

B. $\displaystyle\int_{-8}^{8} \frac{\tan x}{1+x^6} \, dx.$

C. $\displaystyle\int_{-32}^{32} x \cos^3 x \, dx.$

D. $\displaystyle\int_{-1}^{1} x^3 \cos x \, dx.$

E. $\displaystyle\int_{2}^{-2} x^3 \sin^2 x \, dx.$

F. $\displaystyle\int_{0}^{2\pi} \sin x \, dx.$

G. $\displaystyle\int_{0}^{2\pi} \sin^4 x \, dx.$

H. $\displaystyle\int_{0}^{2\pi} \sin^5 x \, dx.$

I. $\displaystyle\int_{\ln 3}^{\ln 3} \frac{\sin(1+e^x)}{x^2+4} \, dx.$

J. $\displaystyle\int_{\sqrt{3}}^{\sqrt{3}} \frac{x^2+\ln x}{1+\sin^2 x} \, dx.$

Solution: Most of these are odd functions with symmetric bounds. The exceptions are the last three several. The ones that are integrals over a point are obviously 0: $\int_a^b f(x)\, dx = 0$ for any function. For the others, you should know that $\int_0^{2\pi} \sin x \, dx = 0$, and similarly $\int_0^{2\pi} \sin^5 x \, dx = 0$. But, $\sin^4 x$ is always positive, and thus the integral can not be equal to zero.

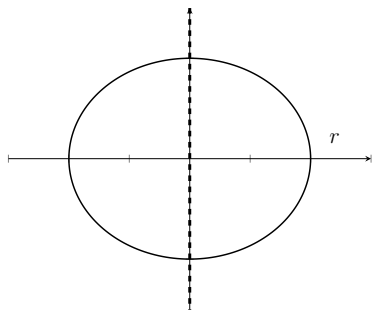
One tricky issue (that we failed to consider when writing the exam) is (B). In this case, the function $f(x) = (\tan x)/(1+x^6)$ has asymptotes at $x = \pm\pi/2$, $\pm 3\pi/2$ and $\pm 5\pi/2$. This integral is an "improper integral", which we haven't discussed yet. In this case, the integrals do not converge and therefore they are technically not equal to 0. We will give credit to students who answered both (B) and (G) or to those who only answered (G).

Written Problem. You will be graded on the readability and reasoning of your work.

15. Consider a circle of radius r centered at the origin, $x^2 + y^2 = r^2$. If this circle is rotated about an axis, it forms a sphere.

    (a) Use the method of **washers/discs** to show the volume V, of a sphere is given by $V = \frac{4}{3}\pi r^3$.

Solution: A sphere is a volume of revolution, where we are revolving the $x^2 + y^2 = r^2$ about the y-axis. For washers, there is no inner radius, but the outer radius is $x = \sqrt{r^2 - y^2}$.



$$\begin{aligned}
V =& \pi \int_{-r}^{r} \left(\sqrt{r^2 - y^2}\right)^2 \, dy \\
=& \pi \int_{-r}^{r} \left(r^2 - y^2\right) \, dy \\
=& \pi \left[r^2 y - \frac{1}{3} y^3\right]_{-r}^{r} \\
=& \pi \left[r^2(r) - \frac{1}{3}(r)^3\right] - \pi \left[r^2(-r) - \frac{1}{3}(-r)^3\right] \\
=& 2\pi \left[r^3 - \frac{1}{3} r^3\right] = \frac{4}{3}\pi r^3
\end{aligned}$$

(b) Use the method of **cylindrical shells** to show the volume V of a sphere is given by $V = \frac{4}{3}\pi r^3$.

Solution: Using shells, the radius of a shell is x and the height of a shell is $2\sqrt{r^2 - x^2}$.

$$\begin{aligned}
V =& 2\pi \int_0^r x \cdot 2\sqrt{r^2 - x^2}\, dx \qquad \text{Use } u = r^2 - x^2 \\
=& 4\pi \int_{x=0}^{x=r} \sqrt{u}\left(-\frac{1}{2}\right) du \\
=& -2\pi \left[\frac{2}{3} \cdot u^{3/2}\right]_{x=0}^{x=r} \\
=& -\frac{4\pi}{3}\left[(r^2 - x^2)^{3/2}\right]_{x=0}^{x=r} \\
=& -\frac{4\pi}{3}\left[0 - r^3\right] \\
=& \frac{4\pi}{3} r^3
\end{aligned}$$

Written Problem. You will be graded on the readability and reasoning of your work.

16. Compute the integrals.

(a) $\int e^x \cos 2x \, dx$.

Solution: We will need to use parts twice, and then use the trick for these integrals.

$$\begin{aligned}\int e^x \cos 2x \, dx =& e^x \left(\frac{1}{2}\right) \sin 2x - \frac{1}{2} \int e^x \sin 2x \, dx \\ & \text{(Parts: } u = e^x,\ dv = \cos 2x \, dx) \\ =& \frac{1}{2} e^x \sin 2x - \frac{1}{2} \left(-\frac{1}{2} e^x \cos 2x + \frac{1}{2} \int e^x \cos 2x \, dx \right) \\ & \text{(Parts: } u = e^x,\ dv = \sin 2x \, dx)\end{aligned}$$

Simplifying gives:

$$\int e^x \cos 2x \, dx = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4} \int e^x \cos 2x \, dx$$

If we let $I = \int e^x \cos 2x \, dx$ then this equation can be written

$$\begin{aligned} I =& \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4} I \\ \frac{5}{4} I =& \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x + C_1 \\ I =& \frac{4}{5} \cdot \left(\frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x + C_1 \right) \\ =& \frac{2}{5} e^x \sin 2x + \frac{1}{5} e^x \cos 2x + C \end{aligned}$$

$$\int e^x \cos 2x \, dx = \frac{2}{5} e^x \sin 2x + \frac{1}{5} e^x \cos 2x + C$$

(b) $\displaystyle\int \sin^2 x \cos^2 x \, dx.$

Solution: Here we use the half angle identities.

$$\begin{aligned}
\int \sin^2 x \cos^2 x \, dx &= \int \left(\frac{1}{2}(1-\cos(2x))\right) \cdot \left(\frac{1}{2}(1+\cos(2x))\right) \, dx \\
&= \frac{1}{4} \int \left(1-\cos^2(2x)\right) \, dx = \frac{1}{4} \int \sin^2(2x) \, dx \\
&= \frac{1}{4} \int \left(\frac{1}{2}(1-\cos 4x)\right) \, dx \\
&= \frac{1}{8} \int (1-\cos 4x) \, dx = \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C \\
&= \frac{1}{8}x - \frac{1}{32}\sin 4x + C
\end{aligned}$$