Section 11.9: Functions as Power Series

- Manipulating known power series (geometric): Substitution, Addition, Multiplication, Division (difficult), Differentiation, Integration

1. **Clicker** (Method: multiplication)

\[
\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots
\]

\[
\frac{1}{2 + x} = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots
\]

Use multiplication of power series to find a series for \( \frac{1}{2+x} \cdot \log(1 + x) \).

This is definitely a pain, but just distribute the terms out.

\[
\frac{1}{2 + x} \cdot \log(1 + x) = \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \right) \left( \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \cdots \right)
\]

(a) \( \sum_{n=1}^{\infty} \frac{1}{2n+1} x^n \)

(b) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^n \)

(c) \( \sum_{n=1}^{\infty} \frac{1}{2n+1} x^n \)

(d) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{2n+1} x^n \)

(e) Something else **Correct**

**Solution:** It is VERY difficult to find a pattern when using the multiplication (or division) technique—don’t even try! Just multiply it out, distribute and get as many terms as you need.

\[
\frac{1}{2 + x} \cdot \log(1 + x) = \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \right) \left( \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \cdots \right)
\]

= \[
\left( \frac{1}{2} - \frac{x}{2} + \frac{5x^2}{12} - \frac{x^3}{3} + \frac{4x^4}{15} - \frac{13x^5}{60} + \frac{151x^6}{840} - \frac{16x^7}{105} + \frac{83x^8}{630} - \frac{73x^9}{630} + \cdots \right)
\]

Note that the proposed answers are the following:

(a) \( \sum_{n=1}^{\infty} \frac{1}{2n+1} x^n = \frac{1}{4} x + \cdots \)

(b) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1} x^n = \frac{1}{4} x + \cdots \)

(c) \( \sum_{n=1}^{\infty} \frac{1}{2n+1} x^n = \frac{1}{4} x + \cdots \)

(d) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{2n+1} x^n = \frac{2}{4} x - \frac{3}{8} n^2 + \cdots \)

(e) Something else **Correct**

None of the answers can be correct!
2. (Method: Long Division–Challenging and worth avoiding whenever possible)
Use long division of power series to find a series for \( \frac{\ln(1+x)}{\ln(1-x)} \)
Note: \( \ln(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \frac{1}{5} x^5 - \ldots \)
and: \( \ln(1 - x) = -x - \frac{1}{2} x^2 - \frac{1}{3} x^3 - \frac{1}{4} x^4 - \frac{1}{5} x^5 - \ldots \)

**Solution:** Tricky! This won’t be tested but basically just use long division (I couldn’t figure out a nice way to typeset the long division so no worked out solution here.)

\[
\frac{\ln(1+x)}{\ln(1-x)} = -1 + x - \frac{1}{2} x^2 + \frac{5}{12} x^3 - \frac{7}{24} x^4 + \frac{191}{720} x^5 - \frac{33}{160} x^6 + \ldots
\]

**Lecture Notes:** Integration of Power Series

**Example:** Power Series for
\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots
\]
\[
\int \frac{1}{1-x} \, dx = \int (1 + x + x^2 + x^3 + x^4 + \ldots) \, dx
\]
\[
-\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots
\]
\[
\ln(1-x) = -C - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots
\]
\[
\ln(1+x) = -C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\]

To find \( C \) we plug in a “good” value of \( x \) to each side. How about \( x = 0 \)? This gives \( C = 0 \). (Note: most of the time \( C \) is equal to 0, but DO NOT fall into the trap of assuming it is always 0!!)

This gives
\[
\ln(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \ldots
\]
\[
= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n
\]

Note that the radius of convergence of this power series is 1 and the interval of convergence is \((-1, 2]\).

3. (Method: Integration. Note \( \int \frac{1}{1+x^2} \, dx = \arctan x + C \))
Find a series for \( \arctan x \).

**Solution:**
\[
\arctan x = \int \frac{1}{1 + x^2} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}
\]
\[
= \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \ldots
\]

Note, you can check to make sure \( C \) is correct by plugging in \( x = 0 \) on both sides of the equation. (This will definitely get you sometimes!)

4. Use your previous series to find a series for \( \arctan(1) = \pi/4 \)
Solution: The series converges for $x = 1$ (why?).

\[
\arctan 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}
\]

\[
= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots
\]

Convergence is terribly slow!

| $n$  | $4S_n$        | $|\pi - 4S_n|$ |
|------|--------------|---------------|
| 10   | 3.232315809405592 | 0.0907   |
| 100  | 3.151493401070999    | 0.0099   |
| 1000 | 3.142591654339543     | 0.000999 |

5. Find a series for $\frac{1}{1+x^\pi}$ and use your series to approximate $\int_0^{0.25} \frac{1}{1 + x^{10}} \, dx$

Solution:

\[
\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots
\]

\[
\frac{1}{1 + x^{10}} = \sum_{n=0}^{\infty} (-1)^n x^{10n} = 1 - x^{10} + x^{20} - x^{30} + x^{40} - x^{50} - x^{60} + \cdots
\]

\[
\int \frac{1}{1 + x^{10}} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1} x^{10n+1} = x - \frac{1}{11} x^{11} + \frac{1}{21} x^{21} - \frac{1}{31} x^{31} + \frac{1}{41} x^{41} - \frac{1}{51} x^{50} - \frac{1}{61} x^{60} + \cdots
\]

\[
\int_0^{1/4} \frac{1}{1 + x^{10}} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1} \left(\frac{1}{4}\right)^{10n+1}
\]

\[
= \frac{1}{4} - \frac{1}{11} \left(\frac{1}{4}\right)^{11} + \frac{1}{21} \left(\frac{1}{4}\right)^{21} - \cdots
\]

Note: no need to find $C$ in the integral since we know we’re going to plug in $x = 0.25$ and $x = 0$. This is an alternating series. So, the integral is equal to $1/4$, accurate to $\frac{1}{11} \cdot \frac{1}{4^{11}} \approx 2 \times 10^{-8}$.

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**Section 11.10: Taylor Series**

- Goal: Given a function: find a power series that equal the function
- Taylor Series
  \[ f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \text{ where } c_n = \frac{f^{(n)}(a)}{n!}. \]
- Maclaurin Series (Taylor Series with $a = 0$)
  \[ f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = \frac{f^{(n)}(0)}{n!}. \]

6. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(a) Find $f(0)$

**Solution:** 1

(b) Find $f(1)$

**Solution:** You can compute partial sums to get an idea of this answer.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$S_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.716666666666667</td>
</tr>
<tr>
<td>10</td>
<td>2.718281801146384</td>
</tr>
<tr>
<td>11</td>
<td>2.718281826198493</td>
</tr>
<tr>
<td>20</td>
<td>2.718281828459045</td>
</tr>
<tr>
<td>100</td>
<td>2.718281828459045</td>
</tr>
<tr>
<td>$e$</td>
<td>2.718281828459045</td>
</tr>
</tbody>
</table>
In any case, $f(1) = e$, which isn’t obvious yet.

(c) Find $f'(x) = f(x)$
(d) Find $f''(x) = f(x)$
(e) Graph $f(x)$.

**Lecture Notes:** You can’t (easily) graph $f(x)$ or answer this question without doing more work. One thing you know, is you’re looking for a function $y = f(x)$ such that $\frac{dy}{dx} = y$ and $y(0) = 1$. This differential equation has a (unique) solution of $y = e^x$ and thus the series is equal to $e^x$. (This method isn’t difficult, but it is beyond what we are doing in calc 2.)

Also, you can’t easily graph a series, but you can graph partial sums, also called Taylor Polynomials.
\( f(x) = e^x \) (Black)

\[
T_0 = 1 \\
T_1 = 1 + x \quad (\text{Blue})
\]

\[
T_2 = 1 + x + \frac{x^2}{2!} \quad (\text{Yellow})
\]

\[
T_3 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad (\text{Green})
\]

\[
T_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}
\]

\[
T_5 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}
\]

\[
T_6 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \quad (\text{Red})
\]

\[
T_7 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}
\]

\[
T_8 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}
\]
Lecture Notes:  The goal now is:

Given a function $f(x)$, find a representation of $f(x)$ with power series.

The idea is similar to:

- Let $f(x) = \frac{1}{1-x}$
- Then $f(x)$ can also be represented as $f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$.

Note: There will always be the question of radius of convergence, interval of convergence but these issue can confuse the main point above.

Idea of attack for a starting function $f(x)$.

- Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$
- Plug in $x = 0$ to get $c_0 = f(0)$
- $\frac{d}{dx}$ gives $f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}$
- Plug in $x = 0$ to get $c_1 = f'(0)$
- $\frac{d}{dx}$ gives $f''(x) = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$
- Plug in $x = 0$ to get $c_2 = f''(0)/2$
- Continue to get $c_n = \frac{f^{(n)}(0)}{n!}$
Do this for $f(x) = e^x$.
I suggest you set up a table to do this:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(0)$</th>
<th>$c_n = f^{(n)}(0)/n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$e^x$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$e^x$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$e^x$</td>
<td>1</td>
<td>$1/2$</td>
</tr>
<tr>
<td>3</td>
<td>$e^x$</td>
<td>1</td>
<td>$1/3!$</td>
</tr>
<tr>
<td>4</td>
<td>$e^x$</td>
<td>1</td>
<td>$1/4!$</td>
</tr>
</tbody>
</table>

Do this for $f(x) = \ln(1 + x)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{(n)}(x)$</th>
<th>$f^{(n)}(0)$</th>
<th>$c_n = f^{(n)}(0)/n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\ln(1 + x)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$1/(1 + x)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$-1/(1 + x)^2$</td>
<td>-1</td>
<td>$1/2$</td>
</tr>
<tr>
<td>3</td>
<td>$2/(1 + x)^3$</td>
<td>2</td>
<td>$2/3!$</td>
</tr>
<tr>
<td>4</td>
<td>$-6/(1 + x)^4$</td>
<td>-3!</td>
<td>$-3!/4! = -1/4$</td>
</tr>
<tr>
<td>5</td>
<td>$4!/(1 + x)^5$</td>
<td>4!</td>
<td>$4!/5! = 1/5$</td>
</tr>
</tbody>
</table>

This is called Taylor Series. Well, actually Maclaurin Series since we centered at $x = 0$ If we center at some point other than 0, it is Taylor Series centered at $x = a$. 
7. Find the Taylor Series for $f(x) = \sin x$ centered at $x = 0$
(Make a table for the derivatives!)

**Solution:**

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots$$

Let's graph some of these Taylor Polynomials.

$T_1$: Red
$T_3$: Blue
$T_5$: Yellow
$T_7$: Green
$T_{11}$: Cyan
$f$: Black

8. Find the Taylor Series for $f(x) = \cos x$ centered at $x = 0$

**Solution:**

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots$$
9. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 0$.

Solution:

$$4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$$

Lets graph some of these Taylor Polynomials.

$T_1$: Red
$T_2$: Blue
$T_3$: Yellow
$T_4$: Green
$T_5 = f$: Black
10. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 1$.

Solution:

$$8 + 16(x - 1) + 26(x - 1)^2 + 17(x - 1)^3 + 2(x - 1)^4 - (x - 1)^5$$

Let's graph some of these Taylor Polynomials.

- $T_1$: Red
- $T_2$: Blue
- $T_3$: Yellow
- $T_4$: Green
- $T_5 = f$: Black