1. [Clicker] Given the sum of the geometric series below:

\[ 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \cdots = \frac{1}{1-x} \]

What is the sum:

\[ 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 + \cdots \]

(a) \( \frac{1}{1-x} \) (b) \( \frac{1}{1+x} \) (c) Correct (d) \( \frac{1}{1-x^2} \) (d) \( \frac{1}{1+x^2} \)

**Solution:** If \( f(x) = \frac{1}{1-x} \) then note the second series is \( f(-x) = \frac{1}{1+x} \).

**Lecture Notes:** Consider the following functions

\[ f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \]

\( T_0(x) = 1 \)

\( T_1(x) = 1 + x \)

\( T_2(x) = 1 + x + x^2 \)

\( T_7(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 \)

Notice the following:

1. The polynomials do a better and better job at approximating the actual function as you take more terms.
2. The series converges on \((-1, 1)\), so we only expect the polynomials to do a good job on this interval.
**Lecture Notes:** Known Power Series: we really only have one big example where we can find the sum:

\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}
\]

In section 11.10, we will learn how to represent any function as a power series. But for now, this is our main example.

Main idea of the section:

Manipulate a known power series to get new power series.

The methods of manipulation are:

- Addition
- Subtraction
- Multiplication (difficult)
- Division (long division, more difficult)
- Differentiation
- Integration

Some of these examples are easy and silly but they illustrate the techniques. (There are often better/easier ways to calculate these series).

Key point is be familiar with these techniques and be able to use them.

\[
\frac{1}{1+x} - \frac{1}{1-x} = \frac{2x}{x^2-1}
\]

\[
\sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} ((-1)^n - 1)x^n
\]

\[
= -2x - 2x^3 - 2x^5 - 2x^7 - 2x^9 = -2 \sum_{n=0}^{\infty} x^{2n+1}
\]

Here’s another way to do the same thing (probably better, definitely more “usual”):

\[
\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \quad \text{(Substitution)}
\]

\[
\frac{2x}{1-x^2} = 2x \sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} 2x^{2n+1}
\]

\[
\frac{2x}{x^2-1} = -2 \sum_{n=0}^{\infty} x^{2n+1}
\]

Differentiation. Note: reindexing the sum is tricky!

Best advice: write out the series term by term and watch for patterns.

\[
\left( \frac{1}{1-x} \right)' = \frac{1}{(1-x)^2}
\]

\[
\frac{1}{(1-x)^2} = \left( \sum_{n=0}^{\infty} x^n \right)' = \sum_{n=0}^{\infty} nx^{n-1}
\]

\[
= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n
\]

2. (Method: Substitution)
Find a series for \( \frac{1}{1+x^2} \)

**Solution:** Letting \( f(x) = \frac{1}{1-x} \) we have \( \frac{1}{1+x^2} = f(-x^2) \):

\[
\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} - x^{12} + \ldots
\]

**Lecture Notes:** Here’s a similar problem, a bit more difficult.

\[
\frac{1}{4+x^2} = \frac{1}{4(1 + x^2/4)} = \frac{1}{4} \cdot \frac{1}{1-(-x^2/4)}
\]

\[
= \frac{1}{4} \sum_{n=0}^{\infty} (-x^2/4)^n = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}
\]

\[
= \frac{1}{4} - \frac{1}{4^2 x^2} + \frac{1}{4^3 x^4} - \frac{1}{4^4 x^6} + \ldots
\]

3. **Clicker** Find a power series for \( f(x) = \frac{x^3}{4+x} \)

(a) \( \sum_{n=0}^{\infty} (-\frac{1}{4})^n x^n \)

(b) \( \sum_{n=0}^{\infty} 4^n x^n \)

(c) \( \sum_{n=0}^{\infty} (-4)^{n+1} x^{n+3} \)

(d) \( \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{n+3} \) [Correct]

(e) There is no series for \( f(x) \).

**Solution:**

\[
\frac{x^3}{4+x} = x^3 \cdot \frac{1/4}{1-(-x/4)}
\]

\[
= \frac{1}{4} x^3 \sum_{n=0}^{\infty} \left( \frac{-x}{4} \right)^n
\]

\[
= \frac{1}{4} x^3 \sum_{n=0}^{\infty} (-\frac{1}{4})^n x^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{n+3}
\]

4. Reindex the solution from the previous question so that the power of \( x \) is \( x^n \).

**Solution:**

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{n+3} = \sum_{n=3}^{\infty} \frac{(-1)^{n-3}}{4^{n-2}} x^n
\]

\[
= 16 \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{4^n} x^n
\]
5. (Method: Differentiation)
Find a series for \( \frac{1}{(1-x)^3} \)

**Solution:**

\[
\frac{1}{(1-x)^3} = \frac{1}{2} \cdot \frac{d^2}{dx^2} \left( \frac{1}{1-x} \right) \\
= \frac{1}{2} \cdot \frac{d^2}{dx^2} \left( \sum_{n=0}^{\infty} x^n \right) \\
= \frac{1}{2} \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{1}{2} \sum_{n=0}^{\infty} (n+2)(n+1)x^n
\]

6. (Method: Integration. Note \( \int \frac{1}{1+x^2} \, dx = \arctan x + C \))
Find a series for \( \arctan x \).

**Solution:**

\[
\arctan x = \int \frac{1}{1+x^2} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \\
=x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots
\]

Note, you can check to make sure \( C \) is correct by plugging in \( x = 0 \) on both sides of the equation. (This will definitely get you sometimes!)

7. Use your previous series to find a series for \( \arctan(1) = \frac{\pi}{4} \)

**Solution:** The series converges for \( x = 1 \) (why?).

\[
\arctan 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \\
= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots
\]

Convergence is terribly slow!

| \( n \) | \( 4S_n \) | \( |\pi - 4S_n| \) |
|-------|---------|----------------|
| 10    | 3.232315809405592 | 0.0907          |
| 100   | 3.151493401070999  | 0.0099          |
| 1000  | 3.142591654339999   | 0.00099         |

8. Find a series for \( \frac{1}{1+x^{10}} \) and use your series to approximate \( \int_{0}^{0.25} \frac{1}{1+x^{10}} \, dx \)

**Solution:**

\[
\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 - x^6 + \cdots \\
\frac{1}{1+ x^{10}} = \sum_{n=0}^{\infty} (-1)^n x^{10n} = 1 - x^{10} + x^{20} - x^{30} + x^{40} - x^{50} - x^{60} + \cdots \\
\int \frac{1}{1+x^{10}} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1} x^{10n+1} = x - \frac{1}{11} x^{11} + \frac{1}{21} x^{21} - \frac{1}{31} x^{31} + \frac{1}{41} x^{41} - \frac{1}{51} x^{51} - \frac{1}{61} x^{61} + \cdots \\
\int_{0}^{1/4} \frac{1}{1+ x^{10}} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1} \left( \frac{1}{4} \right)^{10n+1} \\
= \frac{1}{4} - \frac{1}{11} \left( \frac{1}{4} \right)^{11} + \frac{1}{21} \left( \frac{1}{4} \right)^{21} - \cdots
\]

Note: no need to find \( C \) in the integral since we know we’re going to plug in \( x = 0.25 \) and \( x = 0 \).

This is an alternating series. So, the integral is equal to 1/4, accurate to \( \frac{1}{4} \cdot \frac{1}{4} \approx 2 \times 10^{-8} \).
9. (Method: multiplication) Use multiplication of power series to find a series for \( \frac{1}{1-x} \cdot \frac{1}{1-x} \).
(This is definitely a pain, but just distribute the terms out.)
\[
(1 + x + x^2 + x^3 + x^4 + \cdots)(1 + x + x^2 + x^3 + x^4 + \cdots) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots
\]

10. (Method: Long Division—Challenging and worth avoiding whenever possible) Use long division of power series to find a series for \( \frac{\ln(1+x)}{\ln(1-x)} \)
Note: \( \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots \)
and: \( \ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \cdots \)

**Solution:** Tricky! This won’t be tested but basically just use long division (I couldn’t figure out a nice way to typeset the long division so no worked out solution here.)
\[
\frac{\ln(1+x)}{\ln(1-x)} = -1 + x - \frac{1}{2}x^2 + \frac{5}{12}x^3 - \frac{7}{24}x^4 + \frac{191}{720}x^5 - \frac{33}{160}x^6 + \cdots
\]

11. Find power series for the following functions (you figure out the method(s) to use!)
(a) \( \frac{1}{4+3x} \)

**Solution:**
\[
\frac{1}{4+3x} = \frac{1/4}{1 + \frac{3}{4}x} = \sum_{n=0}^{\infty} \frac{1}{4} \left( -\frac{3x}{4} \right)^n
\]

(b) \( \frac{x}{(1+x)^2} \)

**Solution:**
\[
\frac{x}{(1+x)^2} = x \cdot D_x \left( -\frac{1}{1+x} \right)
\]

(c) \( \frac{1+2x}{1-x} \)

**Solution:**
\[
\frac{1+2x}{1-x} = \frac{1}{1-x} + 2x \left( \frac{1}{1-x} \right)
\]