Section 11.6: Absolute Convergence

- **Definition:** \( \sum a_n \) is **absolutely convergent** if \( \sum |a_n| \) converges.
- **Definition:** \( \sum a_n \) is **conditionally convergent** if \( \sum a_n \) converges but \( \sum |a_n| \) diverges.
- **Theorem:** If \( \sum a_n \) is absolutely convergent then \( \sum a_n \) is convergent.
- **Rearrangements:** Don’t rearrange conditionally convergent series.

Section 11.6: Ratio and Root Tests

- **Ratio Test:** Let \( L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \)
  - If \( L < 1 \) then \( \sum a_n \) converges absolutely
  - If \( L > 1 \) then \( \sum a_n \) diverges
  - If \( L = 1 \) then Ratio Test is inconclusive
- **Root Test:** Let \( L = \lim_{n \to \infty} \sqrt[n]{|a_n|} \)
  - If \( L < 1 \) then \( \sum a_n \) converges absolutely
  - If \( L > 1 \) then \( \sum a_n \) diverges
  - If \( L = 1 \) then Root Test is inconclusive

1. **Clicker** Select the series that converge and those that diverge.

   - A: \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) **Correct**
   - B: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \) **Correct**
   - C: \( \sum_{n=1}^{\infty} \frac{1}{n} \) **Correct**
   - D: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) **Correct**
   - E: \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)
   - F: \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) **Correct**

   How many of the above converge? (a) 2 (b) 3 (c) 4 (Correct) (d) 5 (e) 6

**Lecture Notes:** Sometimes negative terms vs always positive terms.

Here are the main points of this idea:

1. If \( \sum |a_n| \) converges then we say that the series \( \sum a_n \) is **Absolutely Convergent**. Here’s an example: \( \sum \frac{(-1)^n}{n^2} \) is absolutely convergent.
2. If \( \sum a_n \) converges but \( \sum |a_n| \) diverges then the series \( \sum a_n \) is **Conditionally Convergent**.
3. **Theorem** If \( \sum a_n \) is absolutely convergent then \( \sum a_n \) converges.

**Example:** Does the series converge absolutely: \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 + 2n + 1} \)

Converging absolutely means that the series \( \sum |a_n| \) converges. So, we need to test the series

\[
\sum_{i=1}^{\infty} \frac{1}{n^2 + 2n + 1}
\]

which converges by a comparison to \( \sum \frac{1}{n^2} \). So the series converges absolutely and the series converges.

2. Are the following absolutely convergent, conditionally convergent, convergent?

   - (a) \( \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 5} \)
     **Solution:** Not convergent—fails alternating convergence test: \( \lim b_n \neq 0 \).
   - (b) \( \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n + 4} \)
Conditionally convergent by alternating series test. To see $b_n$ is decreasing, look at $f(x) = \sqrt{x+4}$. $f'(x) = \frac{4-x}{2\sqrt{x+4}}$ and is negative when $x > 4$.

(c) $\sum_{n=2}^{\infty} \cos(\pi n) \frac{1}{\sqrt{n}}$

**Solution:** Converges conditionally by alternating series test.

(d) $\sum_{n=1}^{\infty} (-1)^n \sin(\pi n)$

**Solution:** First convince yourself it is alternating. For $n > 0$, we have $0 < \pi/n < \pi$ and thus $\sin(\pi/n) > 0$. And, $\lim \sin(\pi/n) = 0$. Finally, for $n \geq 2$, $\sin(\pi/n)$ is decreasing. Thus the series converges.

But, is the series $\sum \sin(\pi/n)$ convergent? This is more difficult but you can do a limit comparison to $\sum \frac{1}{n}$ to see that it diverges.

(e) $\sum_{n=1}^{\infty} (-1)^n \ln \frac{n}{n}$

**Solution:** $b_n = \ln \frac{n}{n}$ which has limit 0. Let $f(x) = \ln \frac{x}{x}$ and $f'(x) = \frac{1-\ln x}{x^2} < 0$ when $x > e$. Thus converges. It does not converge absolutely.

3. **Clicker** The series $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ converges (why?). What about the following:

$$\sum_{n=0}^{\infty} n \left(\frac{2}{3}\right)^n, \quad \sum_{n=0}^{\infty} n^2 \left(\frac{2}{3}\right)^n, \quad \sum_{n=0}^{\infty} n^3 \left(\frac{2}{3}\right)^n, \quad \sum_{n=0}^{\infty} n^{10000} \left(\frac{2}{3}\right)^n$$

How many converge? (a) 0 (b) 1 (c) 2 (d) 3 (e) 4 [Correct]

**Solution:** All converge. How do we know? Best reason is the ratio test but if you don’t know that, then perhaps you can reason that $(2/3)^n$ eventually overtakes $n^p$ for any power $p$.

4. Let $\sum_{n=0}^{\infty} ar^n$ be a geometric series (so $a_n = ar^n$).

(a) Compute: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r$

(b) Compute: $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} a^{1/n} r = r$

**Lecture Notes:** The previous two problems illustrate the idea of the ratio test, which will be our bread and butter for a while when working with series. The idea is that if the limit of $a_{n+1}/a_n$ is $L$ then for large $n$, the series behaves like a geometric series. And, if the $L < 1$ then the series behaves like a convergent geometric series. The idea of the root test is exactly the same–compare it to a geometric series.

**Technique:**

- **Ratio Test:** Let $L = \lim_{n \to \infty} |a_{n+1}/a_n|$
  - If $L < 1$ then $\sum a_n$ converges absolutely
  - If $L > 1$ then $\sum a_n$ diverges
  - If $L = 1$ then Ratio Test is inconclusive
- **Root Test:** Let $L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$
  - If $L < 1$ then $\sum a_n$ converges absolutely
  - If $L > 1$ then $\sum a_n$ diverges
  - If $L = 1$ then Root Test is inconclusive

5. Converge or diverge (use the ratio test)
(a) \( \sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} \)
Solution: \( \frac{a_{n+1}}{a_n} \to \frac{2}{3} < 1 \) so converges.

(b) \( \sum_{n=0}^{\infty} \frac{1}{n^2} \)
Solution: \( \frac{a_{n+1}}{a_n} \to \frac{(n+1)^2}{n^2} = 1 \) so inconclusive. But we know this converges!

(c) \( \sum_{n=0}^{\infty} \frac{n^3}{(n+1)!} \)
Solution: \( \frac{a_{n+1}}{a_n} \to \frac{(n+1)^3}{n(n+2)} = 0 \) so converges.

(d) \( \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \)
Solution: \( \frac{a_{n+1}}{a_n} = \frac{2(n+1)}{n+1} \to 4 \) so diverges.

(e) \( \sum_{n=1}^{\infty} \frac{4(n!)^2}{(2n)!} \)
Solution: \( \frac{a_{n+1}}{a_n} = \frac{2(n+1)}{2n+1} \to 1 \) so inconclusive.
BUT, note that since \( a_{n+1}/a_n = \frac{2n+2}{2n+1} > 1 \) we have \( a_{n+1} > a_n \). Also, \( a_1 = 2 \), thus \( \lim a_n \neq 0 \). Diverges.

6. Converge or diverge (use the root test)

(a) \( \sum_{n=1}^{\infty} \left( \frac{5n - 3n^3}{7n^3 + 2} \right)^n \)
Solution: \( \lim \sqrt[n]{|a_n|} = \frac{3}{7} < 1 \) so converges.

(b) \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)
Solution: \( \lim \sqrt[n]{|a_n|} = 1 \) so inconclusive.

(c) \( \sum_{n=1}^{\infty} \left( \frac{3}{n + 1} \right)^n \)
Solution: \( \lim \sqrt[n]{|a_n|} = 0 < 1 \) so converges.

(d) \( \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{2n}{n + 1} \right)^n \)
Solution: \( \lim \sqrt[n]{|a_n|} = 2 \) so diverges.

(e) \( \sum_{n=1}^{\infty} \left( \frac{\ln n}{n} \right)^n \)
Solution: \( \lim \sqrt[n]{|a_n|} = 0 \) so converges.

7. Converge or diverge

(a) \( \sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n + 1)} \)
Solution: Ratio test limit gives \( L = 10/16 \), so converges.

(b) \( \sum_{n=1}^{\infty} \left( \frac{5n - 3n^3}{7n^3 + 2} \right)^n \)
Solution: Root test gives \( L = 3/7 \), so converges.

(c) \( \sum_{n=0}^{\infty} \frac{n!}{5^n} \)
Solution: Ratio test limit gives \( L = \infty \), so diverges.

(d) \( \sum_{n=1}^{\infty} \frac{n^n}{3^{2n+1}} \)
Solution: Root test gives \( L = \infty \), so diverges.
(e) \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + 1} \]

**Solution:** Ratio test limit gives \( L = 1 \), so inconclusive. Root test is also not helpful. This converges by alternating series test.

**Lecture Notes: Rearrangements**

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2 \]

Now take positive terms in blocks of one followed by negative terms in blocks of 2. Note that everything is still added in. We then regroup, as below:

\[
1 - \frac{1}{2} - \frac{1}{4} + \frac{3}{6} - \frac{1}{8} + \frac{5}{10} - \frac{1}{12} + \frac{7}{14} - \frac{1}{16} + \frac{9}{18} - \frac{1}{20} + \frac{11}{22} - \frac{1}{24} + \cdots
\]

\[
= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} + \left(\frac{1}{9} - \frac{1}{18}\right) - \frac{1}{20} + \left(\frac{1}{11} - \frac{1}{22}\right) - \frac{1}{24} + \cdots
\]

\[
= \frac{1}{2} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \frac{1}{18} - \frac{1}{20} + \frac{1}{22} - \frac{1}{24} + \cdots
\]

\[
= \frac{1}{2} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots\right) = \frac{1}{2} \ln 2
\]

The points the keep in mind are the following:

- Go ahead and rearrange absolutely convergent series.
- Don’t rearrange conditionally convergent series. In fact, it is possible to rearrange conditionally convergent series to give you any number you want.