# Math 132 - November 20, 2017
**Solutions**

**Section 11.6: Absolute Convergence**

- Definition: $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.
- Definition: $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.
- **Theorem:** If $\sum a_n$ is absolutely convergent then $\sum a_n$ is convergent.
- **Rearrangements:** Don't rearrange conditionally convergent series.

**Section 11.6: Ratio and Root Tests**

- Ratio Test: Let $L = \lim_{n\to\infty} |a_{n+1}/a_n|$
  - If $L < 1$ then $\sum a_n$ converges absolutely
  - If $L > 1$ then $\sum a_n$ diverges
  - If $L = 1$ then Ratio Test is inconclusive
- Root Test: Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$
  - If $L < 1$ then $\sum a_n$ converges absolutely
  - If $L > 1$ then $\sum a_n$ diverges
  - If $L = 1$ then Root Test is inconclusive

1. $\boxed{\textbf{Clicker}}$ Select the series that converge and those that diverge.

   A: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\boxed{\textbf{Correct}}$ B: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ $\boxed{\textbf{Correct}}$ C: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ D: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\boxed{\textbf{Correct}}$ E: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ F: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\boxed{\textbf{Correct}}$

   How many of the above converge? (a) 2 (b) 3 (c) 4 $\boxed{\textbf{Correct}}$ (d) 5 (e) 6

## Lecture Notes: **Sometimes negative terms vs always positive terms.**

Here are the main points of this idea:

1. If $\sum |a_n|$ converges then we say that the series $\sum a_n$ is **Absolutely Convergent**. Here's an example: $\displaystyle\sum \frac{(-1)^n}{n^2}$ is absolutely convergent.
2. If $\sum a_n$ converges but $\sum |a_n|$ diverges then the series $\sum a_n$ is **Conditionally Convergent.**
3. $\boxed{\text{Theorem}}$ If $\sum a_n$ is absolutely convergent then $\sum a_n$ converges.

**Example:** Does the series converge absolutely: $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{n+1}}{n^2+2n+1}$

Converging absolutely means that the series $\sum |a_n|$ converges. So, we need to test the series

$$\sum_{i=1}^{\infty} \frac{1}{n^2+2n+1}$$

which converges by a comparison to $\sum \frac{1}{n^2}$. So the series converges absoltely and the series converges.

2. Are the following absolutely convergent, conditionally convergent, convergent?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$

   **Solution:** Not convergent–fails alternating convergence test: $\lim b_n \neq 0$.

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$

Conditionally convergent by alternating series test. To see $b_n$ is decreasing, look at $f(x) = \frac{\sqrt{x}}{x+4}$. $f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2}$ and is negative when $x > 4$.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$

**Solution:** Converges conditionally by alternating series test.

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin(\pi/n)$

**Solution:** First convince yourself it is alternating. For $n > 0$, we have $0 < \pi/n < \pi$ and thus $\sin(\pi/n) > 0$. And, $\lim \sin(\pi/n) = 0$. Finally, for $n \geq 2$, $\sin(\pi/n)$ is decreasing. Thus the series converges.

But, is the series $\sum \sin(\pi/n)$ convergent? This is more difficult but you can do a limit comparison to $\sum \frac{1}{n}$ to see that it divergses.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$

**Solution:** $b_n = \frac{\ln n}{n}$ which has limit 0. Let $f(x) = \frac{\ln x}{x}$ and $f'(x) = \frac{1-\ln x}{x^2} < 0$ when $x > e$. Thus converges.

It does not converge absolutely.

3. $\boxed{\textbf{Clicker}}$ The series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ converges (why?). What about the following:

$$\sum_{n=0}^{\infty} n \left(\frac{2}{3}\right)^n, \qquad \sum_{n=0}^{\infty} n^2 \left(\frac{2}{3}\right)^n, \qquad \sum_{n=0}^{\infty} n^3 \left(\frac{2}{3}\right)^n, \qquad \sum_{n=0}^{\infty} n^{10000} \left(\frac{2}{3}\right)^n$$

How many converge? (a) 0 (b) 1 (c) 2 (d) 3 (e) 4 $\boxed{\textbf{Correct}}$

**Solution:** All converge. How do we know?? Best reason is the ratio test but if you don't know that, then perhaps you can reason that $(2/3)^n$ eventually overtakes $n^p$ for any power $p$.

4. Let $\sum_{n=0}^{\infty} ar^n$ be a geometric series (so $a_n = ar^n$).

(a) Compute: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$

(b) Compute: $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} a^{1/n} r = r$

**Lecture Notes:** The previous two problems illustrate the idea of the ratio test, which will be our bread and butter for a while when working with series. The idea is that if the limit of $a_{n+1}/a_n$ is $L$ then for large $n$, the series behaves like a geometric series. And, if the $L < 1$ then the series behaves like a convergent geometric series. The idea of the root test is exactly the same–compare it to a geometric series.

**Technique:**

- Ratio Test: Let $L = \displaystyle\lim_{n\to\infty} |a_{n+1}/a_n|$
  - If $L < 1$ then $\sum a_n$ converges absolutely
  - If $L > 1$ then $\sum a_n$ diverges
  - If $L = 1$ then Ratio Test is inconclusive
- Root Test: Let $L = \displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|}$
  - If $L < 1$ then $\sum a_n$ converges absolutely
  - If $L > 1$ then $\sum a_n$ diverges
  - If $L = 1$ then Root Test is inconclusive

5. Converge or diverge (use the ratio test)

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

**Solution:** $\frac{a_{n+1}}{a_n} \to \frac{2}{3} < 1$ so converges.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2}$

**Solution:** $\frac{a_{n+1}}{a_n} \to \frac{(n+1)^2}{n^2} = 1$ so inconclusive. But we know this converges!

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{(n+1)!}$

**Solution:** $\frac{a_{n+1}}{a_n} \to \frac{(n+1)^3}{n(n+2)} = 0$ so converges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

**Solution:** $\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{(n+1)^2} \to 4 > 1$ so diverges.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n(n!)^2}{(2n)!}$

**Solution:** $\frac{a_{n+1}}{a_n} = \frac{2(n+1)}{2n+1} \to 1$ so inconclusive.

BUT, note that since $a_{n+1}/a_n = \frac{2n+2}{2n+1} > 1$ we have $a_{n+1} > a_n$. Also, $a_1 = 2$, thus $\lim a_n \neq 0$. Diverges.

6. Converge or diverge (use the root test)

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$

**Solution:** $\lim \sqrt[n]{|a_n|} = \frac{3}{7} < 1$ so converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

**Solution:** $\lim \sqrt[n]{|a_n|} = 1$ so inconclusive.

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n+1}\right)^n$

**Solution:** $\lim \sqrt[n]{|a_n|} = 0 < 1$ so converges.

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{2n}{n+1}\right)^n$

**Solution:** $\lim \sqrt[n]{|a_n|} = 2$ so diverges.

(e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

**Solution:** $\lim \sqrt[n]{|a_n|} = 0$ so converges.

7. Converge or diverge

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

**Solution:** Ratio test limit gives $L = 10/16$, so converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$

**Solution:** Root test gives $L = 3/7$, so converges.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{5^n}$

**Solution:** Ratio test limit gives $L = \infty$, so diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{2n+1}}$

**Solution:** Root test gives $L = \infty$, so diverges.

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^1+1}$

**Solution:** Ratio test limit gives $L = 1$, so inconclusive. Root test is also not helpful. This converges by alternating series test.

## Lecture Notes: Rearrangements

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$$

Now take positive terms in blocks of one followed by negative terms in blocks of 2. Note that everything is still added in. We then regroup, as below:

$$\begin{aligned}
&1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \frac{1}{9} - \frac{1}{18} - \frac{1}{20} + \frac{1}{11} - \frac{1}{22} - \frac{1}{24} + \cdots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} + \left(\frac{1}{9} - \frac{1}{18}\right) - \frac{1}{20} + \left(\frac{1}{11} - \frac{1}{22}\right) - \frac{1}{24} + \cdots \\
&= \left(\frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{14}\right) - \frac{1}{16} + \left(\frac{1}{18}\right) - \frac{1}{20} + \left(\frac{1}{22}\right) - \frac{1}{24} + \cdots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots\right) = \frac{1}{2}\ln 2
\end{aligned}$$

The points the keep in mind are the following:

- Go ahead and rearrange absolutely convergent series.
- Don't rearrange conditionally convergent series. In fact, it is possible to rearrange conditionally convergent series to give you any number you want.