Math 132 - November 6, 2017
Solutions

11.2: Series

- An infinite sum is the limits of the partial sums:
$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\sum_{i=1}^{n} a_i\right)$$
- Geometric Series I: $s_N = \sum_{n=0}^{N} ar^n = \frac{a(1-r^N)}{1-r}$
- Geometric Series II: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$
- Telescoping Series
- Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$
- Test for Divergence:
  If $\lim a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges

  (No conclusions if $\lim a_n = 0$)

Warm-up Problems

1. $\boxed{\textbf{Clicker}}$ Determine if the following sequence is monotonic and/or bounded: $a_n = \frac{5n-2}{7n+3}$.

   (a) Not monotonic, not bounded

   (b) Monotonic, not bounded

   (c) Bounded, but not monotonic

   (d) Bounded and monotonic

   Solution: Increasing since $f'(n) = 29/(7n+3)^2 > 0$. Bounded since $0 < a_n < 5/7$.

2. Experiment with the following recursive sequences. In particular, write down a few tersm, determine (if possible): monotonicity (increasing/decreasing), bounded (above/below), convergence (converge/diverge, what it converges to) and find an explicit formula for the terms.

   (a) $a_1 = 4$, $a_n = a_{n-1} + 5$
   Solution: This is an *arithmetic* sequence.
   $4, 9, 14, 19, \ldots$.
   $a_n = -1 + 5n$
   Increasing, bounded below, unbounded above, diverges

   (b) $a_1 = 16$, $a_n = \frac{1}{2}a_{n-1}$
   Solution: This is a *geometric* sequence.
   $16, 8, 4, 2, 1, \frac{1}{2}, \ldots$
   $a_n = \frac{32}{2^n}$
   Decreasing, bounded, converges to 0.

   (c) $a_1 = 1$, $a_n = \frac{1}{2}(a_{n-1} + 6)$
   Solution: $1, 2.5, 4.75, 5.375, 5.6875, 5.84375, 5.921875, \ldots$.
   If $L = \lim a_n$ then $L = \frac{1}{2}(L+6)$. Solving gives $L = 6$.
   This doesn't show that the limit exists. Rather, this shows that if there is a limit, then it has to be equal to 6. To see the limit exists, we use mathematical induction to show that the sequence is bounded and increasing.
   Bounded, $a_1 < 6$. If $a_n < 6$ then $a_{n+1} = \frac{1}{2}(a_n + 6) < \frac{1}{2}(6+6) = 6$
   Increasing, true that $a_2 > a_1$. Suppose we know that $a_{k+1} > a_k$. Then, adding six and dividing by 2 gives $\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$, which is $a_{k+2} > a_{k+1}$.

   (d) $a_1 = 0$, $a_2 = 1$, $a_{n+1} = a_n + 2a_{n-1}$
   Solution: $0, 1, 1, 3, 5, 11, 21, 43, \ldots$
   Increasing, unbounded, diverges,

   (e) For the sequence from Probelm 2d, find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$
   Solution: Note that this sequence is undef, $1, 3, \frac{5}{2}, \frac{11}{5}, \ldots$.

And, $\dfrac{a_{n+1}}{a_n} = \dfrac{a_n + 2a_{n-1}}{a_n} = 1 + \dfrac{2}{a_n/a_{n-1}}$ Thus, if $L = \lim a_{n+1}/a_n$ then $L = 1 + 2/L$ and $L^2 = L + 2$. $L^2 - L - 2 = (L-2)(L+1)$ and we must have $L = 2$.

(Can you find a way to make $L = -1$?)

As with previous question, this only shows that if the limit exists, it must be equal to 2. To see that the limit exists, more work is needed (much more difficult!).

3. For a fixed r (real number) and fixed n (positive integer), use long division to simplify

$$\frac{1-r^n}{1-r} =$$

Solution:

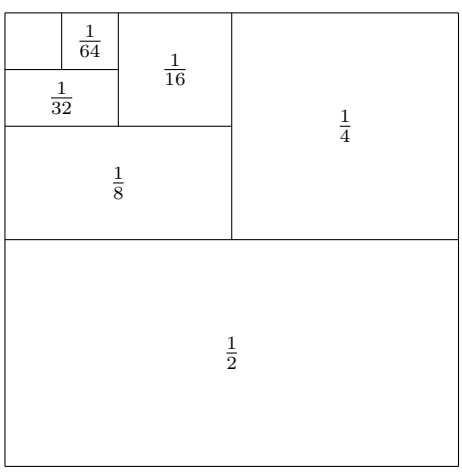
$$\frac{1-r^n}{1-r} = 1 + r + r^2 + \cdots + r^{n-1}$$

Class Problems

4. Find the limit of the following sequence:
$\frac{1}{2}$,
$\frac{1}{2} + \frac{1}{4}$,
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$,
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$,
$\cdots$

The sequence is given by $s_n = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^n$

Solution: First note that s_n is monotonic increasing. If it is also bounded then it converges. The following picture of a 1×1 square should convince you of all this. And, it should also convince you that the sum is equal to 1.



Lecture Notes: This problem shows some of the main ideas about series:

1. A series is an infinite sum of numbers, usually given by a formula.
2. To find the sum of an infinite series, create a sequence of partial sums. (usually/often this sequence of partial sums is just an idea, you don't actually write it down).
3. If the limit of the partial sums converges, we say that series converges and is equal to this limit. Otherwise, we say that the series diverges.
4. Difficulty 1: Understanding what this means
5. Difficulty 2: Determining if the series converges or diverges
6. Difficulty 3: Finding what it converges to (if it converges).

Lecture Notes: **Geometric Series** Here we start with a geometric sequence, which means you multiply each term by a common number to get the next number. This multiply number is called the "common ratio" and is usually denoted by r. Here's the example from the previous problem:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

Note that $a_n = \frac{1}{2}a_{n-1}$ or $a_n = \frac{1}{2}^n$. We can add these all up and form the partial sums and find the sum

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

In general we have

Geometric Sequence: $a, ar, ar^2, ar^3, \ldots$

Geometric Series: $a + ar + ar^2 + ar^3 + \cdots$

Writing this in more concise form:

$$S = \sum_{n=0}^{\infty} ar^n$$

And the partial sums of a geometric series are

$$s_N = \sum_{n=0}^{N-1} ar^n = a\left(1 + r + r^2 + \cdots + r^{N-1}\right)$$

We can write this with our long division done above

$$s_N = \sum_{n=0}^{N-1} ar^n = \frac{a(1-r^N)}{1-r}$$

And we can find the limit of partial sums, which converges only if $|r| < 1$. If $|r| < 1$ then

$$\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} \frac{a(1-r^N)}{1-r} = \frac{1}{1-r}$$

Example:

$$4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots = \sum_{n=0}^{\infty} 4\left(\frac{1}{3}\right)^n = \frac{4}{1-(1/3)} = 6$$

The keys, to me, are

1. Make sure $|r| < 1$, otherwise diverges
2. Find r and find a, you don't usually have to write it in sum form, but that is sometimes useful too.

5. $\boxed{\textbf{Clicker}}$ Find $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

   (a) 1 (b) 2 $\boxed{\textbf{Correct}}$ (c) 3 (d) 4

   Solution: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{(2/3)}{1-(2/3)} = 2$

6. Use Geometric series to compute the infinite sum:

   (a) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n = \frac{4}{3}$

(b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n = \text{Diverges!}$

(c) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2}{3}\right)^{n-2} = 3$

(d) $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n = \frac{1}{1+2/3} = \frac{3}{5}$

(e) $\displaystyle\sum_{n=2}^{\infty}\frac{2^n\cdot 3^{n-3}}{5^{2n+3}} = \sum_{n=2}^{\infty}\left(\frac{3^{-3}}{5^3}\right)\left(\frac{2\cdot 3}{25}\right)^n = \left(\frac{1}{9\cdot 125}\right)\left(\frac{6}{25}\right)^2\left(\frac{1}{1-6/25}\right)$

Lecture Notes: You can, of course, also use the geometric series formulas to compute partials sums. This is sometimes useful. In anycase, you do have to be careful about what the terms in the formula mean. For me:

1. a is the first term
2. r is the common ratio
3. n is the number of terms

I like writing these formulas as:

$$s_N = \sum_{n=0}^{N-1} ar^n = \frac{a(1-r^n)}{1-r}$$
$$S = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

The book, instead, writes the following:

$$s_N = \sum_{n=1}^{N} ar^{n-1} = \frac{a(1-r^n)}{1-r}$$
$$S = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

but in either case, the interpretation of a, r and N are the same.

7. Use Geometric series to compute the partial sums

(a) $\displaystyle\sum_{n=1}^{4}\left(\frac{2}{3}\right)^n = \frac{(2/3)\cdot(1-(2/3)^4)}{1-(2/3)}$

(b) $\displaystyle\sum_{n=1}^{4}\left(\frac{2}{3}\right)^{n-1} = \frac{(1)\cdot(1-(2/3)^4)}{1-(2/3)}$

(c) $\displaystyle\sum_{n=4}^{7}\left(\frac{2}{3}\right)^n = \frac{(2/3)^4\cdot(1-(2/3)^4)}{1-(2/3)}$

Lecture Notes: **Telescoping Series:** Here you start with an algebraic expression and use partial fractions and then sometimes something amazing happens (terms all cancel out).

Example:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots$$

If you look at it a while you'll see lots of terms canceling and $s_N = 1 - \frac{1}{N+1}$ Thus, $\lim s_N = 1$

For telescoping series:

1. See if partial fractions applies
2. Write down some terms and see what cancels out
3. Write down a formula for s_N.
4. Take the limit of s_N. This is important because things might look like they cancel but the limit diverges.

8. Use telescoping series to compute the series.

(a) $$\boxed{\sum_{n=2}^{\infty} \frac{1}{n^2 - n} = \sum_{n=1}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right)}$$

$$s_N = 1 - \frac{1}{N}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - n} = 1$$

(b) $$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 3} = \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{n+1} - \frac{1}{n+3} \right)}$$

$$s_N = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 3} = \frac{5}{12}$$

(c) $$\boxed{\sum_{n=1}^{\infty} \ln \left(\frac{k+1}{k} \right) = \sum_{n=1}^{\infty} (\ln(k+1) - \ln k)}$$

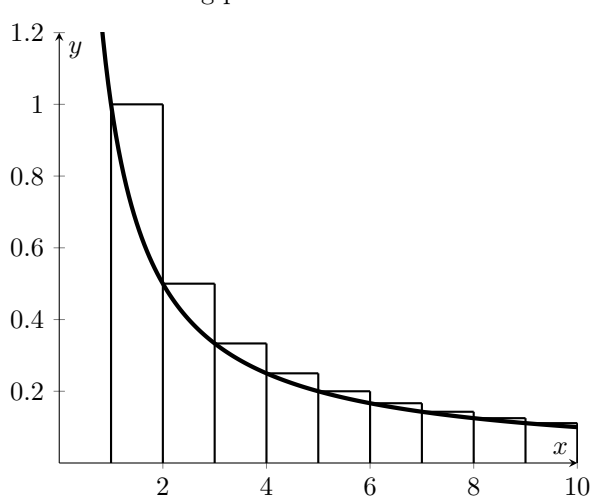
$$s_N = \ln n - \ln 1$$

$$\sum_{n=1}^{\infty} \ln \left(\frac{k+1}{k} \right) = \text{Diverges}$$

Lecture Notes: **Harmonic Series** Here we investigate

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Note the following picture.



The point is that

$$\sum_{n=0}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x} = \infty$$

and thus $\sum \frac{1}{n}$ diverges!

Lecture Notes: A test for divergence: Here are some examples to understand:

$$\sum_{n=1}^{\infty} 1 = \infty$$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \infty$$

$$\sum_{n=1}^{\infty} (-1)^n = DNE$$

Some things for sure about $\sum_{n=1}^{\infty} a_n$ are the following:

1. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ must diverge
2. If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ might diverge or might converge.