11.1 Sequences

Terms
- Increasing sequence
- Decreasing Sequence
- Monotonic Sequence
- Bounded Above Sequence
- Bounded Below Sequence
- Bounded Sequence

Monotonic Sequence Theorem:
Every bounded monotonic sequence converges.

Warm-up Problems

1. **Clicker** Let \( \{a_n\} \) be the sequence such that \( a_n = \frac{3n}{n+5} \). Find a number \( M \) such that \( a_n \leq M \) for all \( n \).
   (a) 0 (b) \( \frac{1}{2} \) (c) 1 (d) 100 (e) Not possible to find such an \( M \)

2. Write the first few terms of the sequence and then find \( \lim_{n \to \infty} a_n \)
   (a) \( a_n = \frac{\ln n}{n} \)
   (b) \( a_n = \sqrt{n} \)
   (c) Let \( x > 0 \) be fixed. Let \( a_n = x^{1/n} \).

3. Write the first few terms of the sequence and then find \( \lim_{n \to \infty} a_n \) where \( a_n = x^n \).

4. Let \( x \) be fixed and let \( a_n = (1 + \frac{x}{n})^n \). Find \( \lim_{n \to \infty} a_n \).

5. Let \( x \) be fixed and let \( a_n = \frac{x^n}{n!} \). Find \( \lim_{n \to \infty} a_n \).

6. Find \( \lim_{n \to \infty} a_n \) where \( a_n = \frac{n!}{n^n} \).

7. Continued Fractions (Fun and mathematically important, but not important for calculus class).
   (a) Find the limit of the sequence:
   \[ a_1 = 1, \quad a_2 = 1 + \frac{1}{1}, \quad a_3 = 1 + \frac{1}{1 + \frac{1}{1}}, \quad a_4 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}, \quad a_5 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}, \text{ etc.} \]
   (b) Find the limit of the sequence:
   \[ a_1 = 2, \quad a_2 = 2 + \frac{1}{1}, \quad a_3 = 2 + \frac{1}{1 + \frac{1}{2}}, \quad a_4 = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}, \quad a_5 = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}, \quad a_6 = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}}, \text{ etc.} \]

In this sequence, the “diagonal” of the fractions follows the pattern: \([2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, \ldots]\)
Class Problems

8. For each of the following sequences, determine if it is bounded (above/below/both), monotonic/increasing/decreasing.

(a) \( a_n = \frac{1}{n} \)
(b) \( a_n = \frac{-1}{n} \)
(c) \( a_n = \frac{(-1)^n}{n} \)
(d) \( a_n = \frac{n + 1}{n} \)
(e) \( a_n = \frac{n + 1}{n + 2} \)
(f) \( a_n = n^2 \)
(g) \( a_n = -n^2 \)
(h) \( a_n = (-1)^nn^2 \)

9. Determine if the following sequence is monotonic and bounded: \( a_n = \frac{5n - 2}{7n + 3} \).

10. Experiment with the following recursive sequences. In particular, write down a few terms, determine (if possible): monotonicity (increasing/decreasing), bounded (above/below), convergence (converge/diverge) and find an explicit formula for the terms.

(a) \( a_1 = 4, a_n = a_{n-1} + 5 \)
(b) \( a_1 = 16, a_n = \frac{1}{2}a_{n-1} \)
(c) \( a_1 = 1, a_n = \frac{1}{2}(a_{n-1} + 6) \)
(d) \( a_1 = 0, a_2 = 1, a_{n+1} = a_n + 2a_{n-1} \)
(e) For the sequence from Problem 10d, find \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \).

11. [Clicker] For a fixed \( r \) (real number) and fixed \( n \) (positive integer), use long division to simplify

\[
\frac{1 - r^n}{1 - r} =
\]

(a) \( r^{n-1} \)
(b) \( 1 + r^{n-1} \)
(c) \( 1 - r^{n-1} \)
(d) \( 1 + r + r^2 + r^3 + \ldots + r^{n-1} \)

12. Find the limit of the following sequence: \( \frac{1}{2} \),
\( \frac{1}{2} + \frac{1}{4} \),
\( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \),
\( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \),
\ldots

The sequence is given by

\[
s_n = \sum_{i=0}^{n} \left( \frac{1}{2} \right)^i
\]