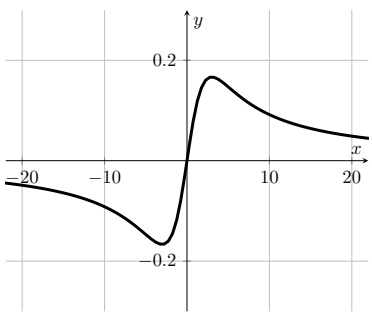
Math 132 - October 27, 2017
Solutions

8.1 Arc Length	**8.2 Surface Area**
$L = \int ds = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$	$S = \int_a^b dS = \int_a^b 2\pi r\, ds = \int_a^b 2\pi r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Warm-up Problems

1. **Clicker**

Below is a graph of $f(x) = \frac{x}{9+x^2}$.



Find $\displaystyle\int_{-\infty}^{\infty} \frac{x}{9+x^2}\, dx$

(a) Converges to 0

(b) Converges to something else

(c) Diverges to $+\infty$

(d) Diverges to $-\infty$

(e) Diverges **Correct**

Note the following work:

$$\int_0^{\infty} \frac{x}{9+x^2}\, dx = \lim_{t\to\infty} \int_0^t \frac{x}{9+x^2}\, dx = \lim_{t\to\infty} \frac{1}{2}\ln(x^2+9)\Big|_0^t$$
$$\lim_{t\to\infty} \frac{1}{2}\ln(t^2+9) - \frac{1}{2}\ln(9) = \lim_{t\to\infty} \frac{1}{2}\left[\ln\left(\frac{t^2+9}{9}\right)\right] = \infty$$

Solution: This example is supposed to shows you what happens if you don't split it up. The basic rule/idea is to split up the integral with each integral with only one improper piece. Then, if any of them diverge then the whole thing diverges (which is what happens here).

NOT CORRECT
$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x}{9+x^2}\, dx &= \lim_{t\to\infty} \int_{-t}^{t} \frac{x}{9+x^2}\, dx \\ &= \lim_{t\to\infty} \frac{1}{2}\ln(x^2+9)\Big|_{-t}^{t} \\ &= \lim_{t\to\infty} \left[\frac{1}{2}\ln(t^2+9) - \frac{1}{2}\ln((-t)^2+9)\right] \\ &= \lim_{t\to\infty} \left[\frac{1}{2}\ln(t^2+9) - \frac{1}{2}\ln(t^2+9)\right] = 0\end{aligned}$$

Instead, you should do it this way:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x}{9+x^2}\, dx &= \int_{-\infty}^{0} \frac{x}{9+x^2}\, dx + \int_0^{\infty} \frac{x}{9+x^2}\, dx \\ &= \lim_{t\to-\infty} \int_t^0 \frac{x}{9+x^2}\, dx + \lim_{t\to\infty} \int_0^t \frac{x}{9+x^2}\, dx \\ &= \lim_{t\to-\infty} \frac{1}{2}\ln(x^2+9)\Big|_t^0 + \lim_{t\to\infty} \frac{1}{2}\ln(x^2+9)\Big|_0^t\end{aligned}$$

and these two limits are infinite. Yes, its true that one is $+\infty$ and the other is $-\infty$, but it is not true that $(+\infty)+(-\infty) = 0$. Rather, $(+\infty) + (-\infty)$ is undefined (or more work is needed to evaluate and in this context, the integral diverges).

Here's one way to think about this (and why the area is not 0). You and a friend start at the origin with radios. One of you walks to the right (towards $+\infty$ on the x-axis) and the other walks to the left (towards $-\infty$ on the y-axis). Periodically, you radio each other up and tell each other the area you have found. You then add your results together (which will be subtracting since one will be negative). You do this forever (taking the limit) of your results.

If the two of you walk the same speed, then the above description gives you:

$$\lim_{t\to\infty} \int_{-t}^{t} \frac{x}{9+x^2}\, dx = 0$$

BUT, if one of you runs twice the speed as the other is walking, then it would look like this:

$$\begin{aligned}
\lim_{t\to\infty} \int_{-t}^{2t} \frac{x}{9+x^2}\, dx &= \lim_{t\to-\infty} \frac{1}{2}\ln(x^2+9)\Big|_{t}^{2t} \\
&= \lim_{t\to-\infty} \frac{1}{2}\ln(4t^2+9) - \frac{1}{2}\ln(t^2+9) \\
&= \lim_{t\to-\infty} \frac{1}{2}\ln\left(\frac{4t^2+9}{t^2+9}\right) = \frac{1}{2}\ln 4
\end{aligned}$$

Similarly, you could run 4 times as fast, or exponentially as fast, etc. All of these different rates would result in different results. Mathematically, it is probably best if the same result can be achieved every time. Thus, if this can not happen, we say the integral diverges.

The net result of this is the following rule:

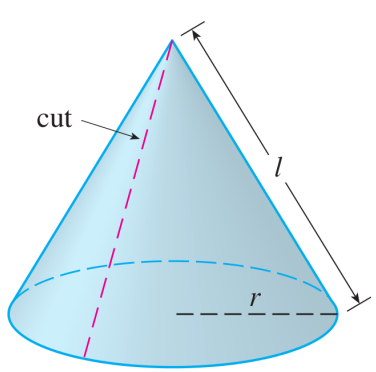
- Split up improper integrals so that each integral has at most one improper piece.
- If any of these split up integrals diverges then the integral diverges.

2. Find the length of the line segment between the points $(x_i, f(x_i))$ and $(x_j, f(x_j))$.
Solution:

$$\begin{aligned}
L =& \sqrt{(x_i - x_j)^2 + (f(x_i) - (x_j))^2} \\
=& \sqrt{(\Delta x)^2 + (\Delta y)^2} \\
=& \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,(\Delta x)
\end{aligned}$$

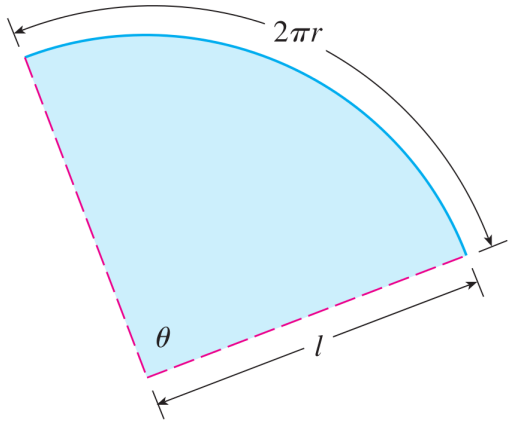
3. **Clicker**

Find the surface area of the cone



(a) $\pi r l$ **Correct**
(b) $2\pi r l$
(c) $\pi(r^2 + l^2)$
(d) $\pi\sqrt{r^2 + l^2}$

Solution: To find this area, you just cut open the cone as indicated:

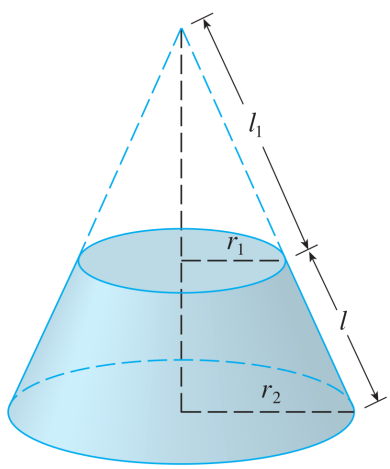


When you do this, you can see that it is a portion of a disk of radius l. The fraction of the disk is $\frac{2\pi l}{2\pi r} = \frac{l}{r}$. Thus, the area is this fraction of the area of the disk:

$$S = \left(\frac{l}{r}\right)\pi r^2 = \pi r l$$

4. **Clicker**

Find the surface area of the frustum of the cone.
(Just the cone part, not the top/bottom.)



(a) $\pi l r_1$

(b) $\pi l r_2$

(c) $2\pi l\left(\frac{r_1 + r_2}{2}\right)$ **Correct**

(d) $2\pi l\left(\frac{r_1 - r_2}{2}\right)$

Solution: Here you just take a large cone minus a small cone:

$$S = (\text{Big Cone}) - (\text{Small Cone}) = \pi(l + l_1)r_2 - \pi l_1 r_1 = \pi\left[(r_2 - r_1)l_1 + r_2 l\right]$$

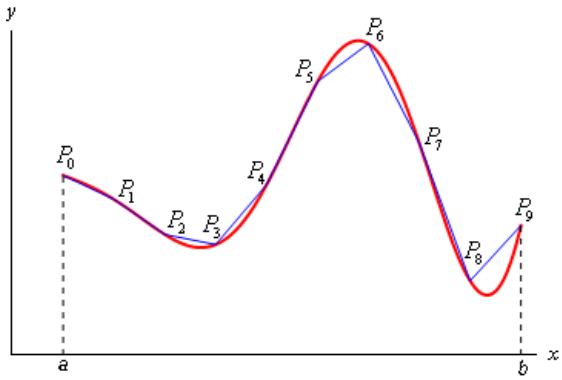
From similar triangles, we have $\frac{l_1}{r_1} = \frac{l+l_1}{r_2}$ which gives $r_2 l_1 = r_1 l + r_1 l_1$ or $(r_2 - r_1)l_1 = r_1 l$.
Putting these together:

$$S = \pi\left[r_1 l + r_2 l\right] = \pi(r_1 + r_2)l = 2\pi l\left(\frac{r_1 + r_2}{2}\right) = 2\pi r l$$

where r is the average radius of the frustum.

Class Problems

Lecture Notes: **Length and area.** Length first.
To find length, just take your curve, chop it up, pretend each segment is a line and then take the number of chops to infinity.



If you do this, you get the following

$$L \approx \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} |P_{i-1}P_i| = \sum_{i=1}^{N} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sum_{i=1}^{N} \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum_{i=1}^{N} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

In the limit this becomes an equality, with derivatives and integrals:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Sometimes this is written

$$L = \int_a^b ds$$

where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$.

The formula is easy, relatively easy to understand and use, but generally the integrals involved are terrible (unless we "cook them up" to be easy).

Here's an example. Length of $y = x^2$ from $x = 0$ to $x = 1$:

$$\begin{aligned}
L &= \int_0^1 \sqrt{1 + 4x^2}\ dx = \int_{x=0}^{x=1} \sqrt{1 + \tan^2\theta} \cdot \frac{1}{2}\sec^2\theta\ d\theta \qquad (x = \frac{1}{2}\tan\theta) \\
&= \frac{1}{2}\int_{x=0}^{x=1} \sec^3\theta d\theta \\
&= \frac{1}{4}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]\Big|_{x=0}^{x=1} \\
&= \frac{1}{4}\left[2x\sqrt{1+4x^2} + \ln|\sqrt{1+4x^2} + 2x|\right]\Big|_{x=0}^{x=1} \\
&= \frac{1}{4}\left[2\sqrt{5} + \ln|\sqrt{5}+2|\right] = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4} \approx 1.478943
\end{aligned}$$

5. Set up arc length integrals

   (a) Find length of $y = 2x + 3$ between $x = 0$ and $x = 2$
   Solution:

   $$L = \int_0^2 \sqrt{1 + 2^2}\ dx = \int_0^2 \sqrt{5}\ dx = 2\sqrt{5}$$

   (b) Find length of $y = \cos x$ between $x = 0$ and $x = \pi$
   Solution:

   $$L = \int_0^\pi \sqrt{1 + \sin^2 x}\ dx = ??$$

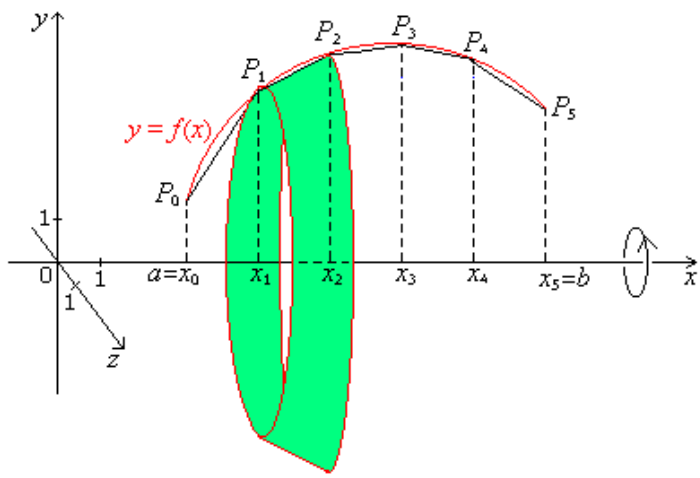
You can compute this using numerical methods to get an approximate of 3.8202 . . . .

(c) Find length of $y = 2(x-1)^{3/2}$ from $x = 1$ to $x = 5$.
Solution:

$$L = \int_1^5 \sqrt{1 + 9(x-1)}\, dx = \frac{2(37^{3/2})}{27} - \frac{2}{27}$$

Lecture Notes: **Surface Area.** We are going to work on surface area of solids of revolution. For more general areas, look ahead to the material in any calc 3 textbook.

The idea is to take a surface of revolution, chop it up, which yeilds small frustum-like rings and add up these areas of frustums.



When you do this, you get

$$\begin{aligned} S &\approx \sum (\text{Area Frustum}) \\ &= \sum 2\pi r l \end{aligned}$$

Taking limits and integrating gives

$$S = \int_a^b 2\pi r\, ds = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

Here's an example.

Find the area of a parabola $y = x^2$ rotated about the y-axis from $x = 0$ to $x = 1$. Draw the picture. You can do this integral either dx or dy

$$\begin{aligned} S &= \int_0^1 2\pi r l\, ds \\ &= \int_0^1 2\pi x\sqrt{1 + (dy/dx)^2}\, dx \\ &= \int_0^1 2\pi x\sqrt{1 + 4x^2}\, dx \\ &= \frac{1}{6}\pi(1 + 4x^2)^{3/2}\Big|_0^1 = \frac{1}{6}\pi(5^{3/2} - 1) \approx 5.3304 \end{aligned}$$

Here's the dy way:

$$\begin{aligned} S &= \int_0^1 2\pi r l \, ds \\ &= \int_0^1 2\pi x \sqrt{1 + (dx/dy)^2} \, dy \\ &= \int_0^1 2\pi \sqrt{y} \sqrt{1 + \frac{1}{4y}} \, dy \\ &= \int_0^1 \pi \sqrt{1 + 4y} \, dy = \frac{1}{6}\pi(5^{3/2} - 1) \end{aligned}$$

6. Set up surface area integrals

   (a) Find the area of $y = \sqrt{x}$ on the interval $[3/4, 15/4]$ rotated about the x-axis.
   Solution:

   $$S = \int_{3/4}^{15/4} 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} \, dx = \frac{28}{3}$$

   (b) Find the area of $x = \frac{y^3}{3}$ on the interval $y = 0$ to $y = 1$ rotated about the y-axis.
   Solution:

   $$S = \int_0^1 2\pi \frac{y^3}{3} \sqrt{1 + y^4} \, dx = \frac{\pi}{9}(2\sqrt{2} - 1)$$

   (c) Find the area of $y = \sin x$ from $x = 0$ to $x = \pi$.
   Solution:

   $$S = \int_0^\pi 2\pi \sin x \sqrt{1 + \cos^2 x} \, dx \approx 2.295587\ldots$$