Math 132 - October 25, 2017
Solutions

7.8 Improper Integrals

- Type I: ∞ as limit of integral
- Type II: Vertical Asymptote (sometimes difficult to recognize)
- Use a limit to resolve
- Use a separate limit for each improper piece if there are multiple improper pieces (see examples)

Warm-up Problems

1. Compute $\lim_{x\to\infty} \frac{\ln x}{x}$

   Solution: Use L'Hopital's Rule.

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln x}{x} &= \lim_{x\to\infty} \frac{1/x}{1} \qquad \text{by L'Hopital}\\ &= \lim_{x\to\infty} \frac{1}{x} = 0\end{aligned}$$

2. Compute $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$

   Solution: Use L'Hopital's Rule.

$$\begin{aligned}\lim_{x\to\infty} \frac{(\ln x)^2}{x} &= \lim_{x\to\infty} \frac{2\cdot(\ln x)\cdot(1/x)}{1} \qquad \text{by L'Hopital}\\ &= \lim_{x\to\infty} \frac{2\ln x}{x}\\ &= \lim_{x\to\infty} \frac{2\cdot(1/x)}{1} \qquad \text{by L'Hopital}\\ &= \lim_{x\to\infty} \frac{2}{x} = 0\end{aligned}$$

3. $\boxed{\textbf{Clicker}}$ Select how many in the list below converge: (a) 1 (b) 2 $\boxed{\textbf{Correct}}$ (c) 3 (d) 4 (e) 5

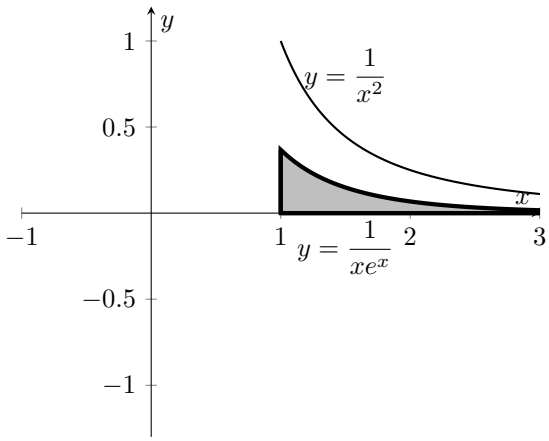
   - $\int_1^\infty \frac{1}{x^3}\,dx$ $\boxed{\textbf{Correct}}$

     Solution:

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^3}\,dx = \lim_{t\to\infty} -\frac{1}{2x^2}\bigg|_1^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2}\right) = \frac{1}{2}$$

   - $\int_1^\infty \frac{1}{xe^x}\,dx$ $\boxed{\textbf{Correct}}$

     Solution: This one is trickier since you won't be able to compute the indefinite integral. Instead, think about the picture and compare to $\frac{1}{x^2}$:



We know that the integral of $1/x^2$ converges and therefore the smaller integral of $1/(xe^x)$ must also converge. Finding its value.... that's another matter.

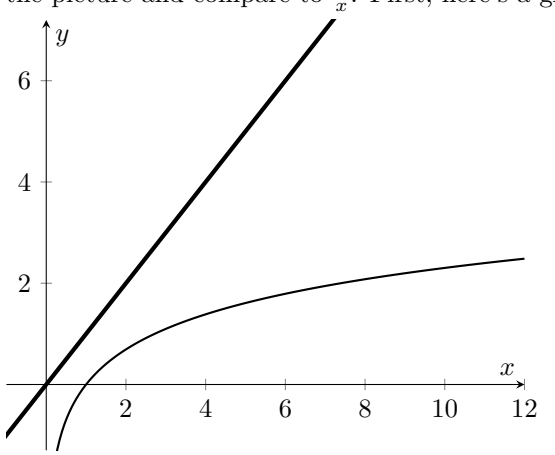
- $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

  Solution:

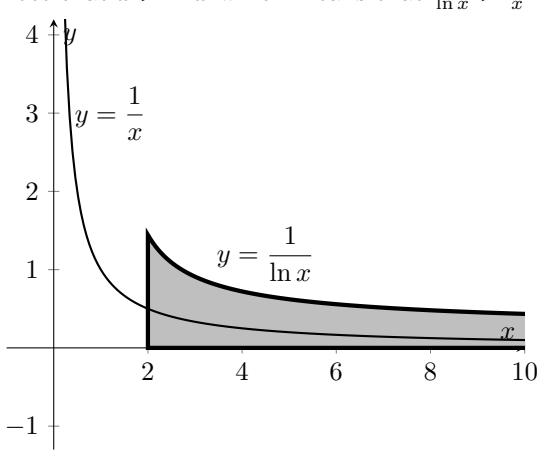
$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x|_1^t = \lim_{t\to\infty} \ln t = \infty$$

- $\displaystyle\int_2^\infty \frac{1}{\ln x}\,dx$

  Solution: This one is trickier since you won't be able to compute the indefinite integral. Instead, think about the picture and compare to $\frac{1}{x}$. First, here's a graph of $y = x$ and $y = \ln x$:



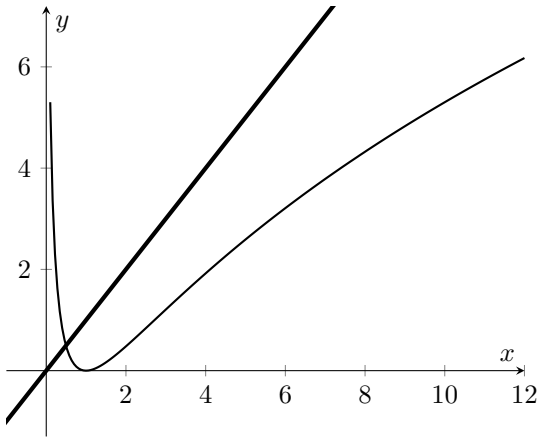
Note that $x > \ln x$ which means that $\frac{1}{\ln x} > \frac{1}{x}$. Now consider the graphs of the integrals $\int_1^\infty \frac{1}{\ln x}\,dx$ and $\int_1^\infty \frac{1}{x}\,dx$:



We know that the integral of $1/x$ diverges and therefore the bigger integral of $1/(\ln x)$ must also diverge.

- $\displaystyle\int_2^\infty \frac{1}{(\ln x)^2}\,dx$

Solution: Another tricy one that we can't find an antiderivative for. Lets graph it with $1/x$ and see what we can figure out.



The graph suggests that $x > (\ln x)^2$. If this is true, then the logic from the previous problem says that $\int_2^\infty \frac{1}{(\ln x)^2}\,dx$ diverges.

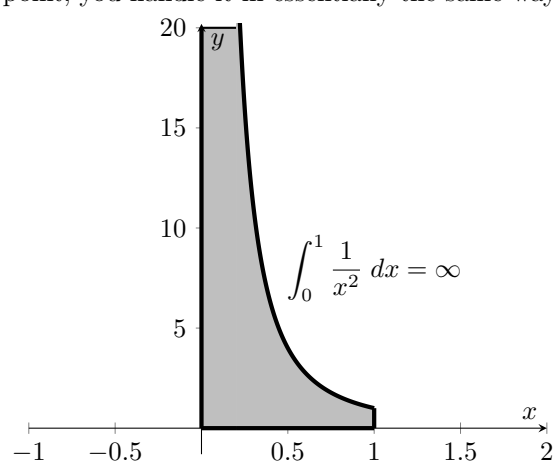
To convince yourself of this note that (by using L'Hopital's rule):

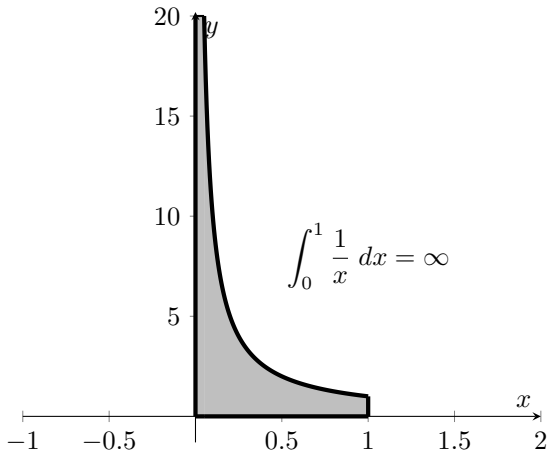
$$\begin{aligned}
\lim_{x\to\infty} \frac{(\ln x)^2}{x} &= \lim_{x\to\infty} \frac{2\cdot(\ln x)\cdot(1/x)}{1} && \text{by L'Hopital} \\
&= \lim_{x\to\infty} \frac{2\ln x}{x} \\
&= \lim_{x\to\infty} \frac{2\cdot(1/x)}{1} && \text{by L'Hopital} \\
&= \lim_{x\to\infty} \frac{2}{x} = 0
\end{aligned}$$

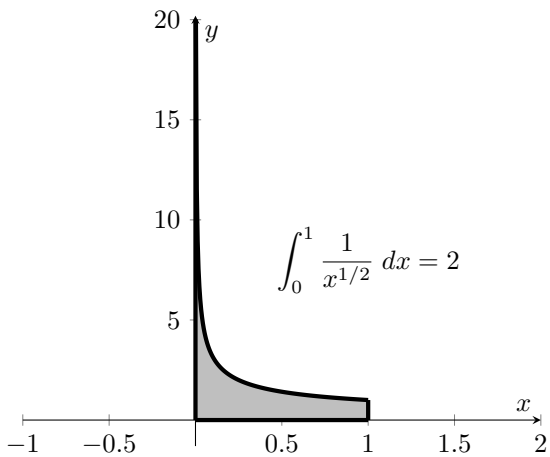
What this means is that for large x, x is so much bigger than $(\ln x)^2$ that eventually $(\ln x)^2$ is negligible compared to x. Thus, our reasoning from the graph is correct.

Class Problems

Lecture Notes: The other type of improper integrals are called "Type II" integrals and again the name isn't important. What is important is to realize that you are integrating a vertical asymptote. Again, once you recognize this point, you handle it in essentially the same way (and, it always helps to graph it).







$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0}\int_t^1 \frac{1}{\sqrt{x}}\,dx$$

Then, you take the integral like usual and take the limit "as usual":

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0}\int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0} 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0} 2\sqrt{1} - 2\sqrt{t} = 2$$

In fact, if you notice in this example that if you "forgot" it was improper and just "plugged in" (without a limit) it would have worked just fine. Sometimes this happens, but sometimes this can get you in trouble. Best to just do it the right way with the limit.

4. **Clicker** For which values of p, does $\int_0^1 \frac{1}{x^p}\,dx$ converge?

   (a) $p < 1$ **Correct**
   (b) $p \leq 1$
   (c) $p > 1$
   (d) $p \geq 1$

Lecture Notes: Again, this is the key example to understand:

If $p > 1$ then

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0}\int_t^0 \frac{1}{x^p}\,dx = \lim_{t\to 0} \frac{1}{(1-p)x^{p-1}}\Big|_t^1 = \lim_{t\to 0}\left(\frac{1}{(1-p)} - \frac{1}{(1-p)t^{p-1}}\right) = \infty$$

If $p = 1$ then

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0}(0 - \ln(t)) = \infty$$

If $0 < p < 1$ then

$$= \lim_{t\to 0}\int_t^0 \frac{1}{x^p}\,dx = \lim_{t\to 0}\left.\frac{x^{1-p}}{1-p}\right|_t^1 = \lim_{t\to 0}\left(\frac{1}{1-p} - \frac{t^{1-p}}{1-p}\right) = \frac{1}{1-p}$$

If $p = 0$ then

$$\int_0^1 \frac{1}{x^p}\,dx = \int_0^1 1\,dx = 1$$

If $p < 0$ then

$$\int_0^1 \frac{1}{x^p}\,dx = \int_0^1 x^{-p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right|_0^1 = \frac{1}{-p+1}$$

It is important to understand this example.

Lecture Notes: One technical point (that I usually write technically wrong). If you are to compute $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ you should use a one-sided limit like this:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left.2\sqrt{x}\right|_t^1 = \lim_{t\to 0^+} 1 - 2\sqrt{t} = 1$$

Of course, ignoring the one-sided limit is technically wrong and it usually does not get you in to trouble. Usually.

5. Determine if the following improper integrals converge. If they converge, determine their value.

(a) $\displaystyle\int_1^\infty xe^{-x}\,dx$
Solution: Converges to $1/e$

(b) $\displaystyle\int_0^\infty \sin x\,dx$
Solution: Diverges because the limit $\lim_{t\to\infty}\cos t$ does not exist

(c) $\displaystyle\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$
Solution: Vertical asymptote at $x = 3$
$= \lim_{t\to 3^-}\int_0^t \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3}(2\sqrt{3} - 2\sqrt{3-t}) = 2\sqrt{3}$

(d) $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$ Vertical asymptote at $x = 3$:
$= \lim_{t\to 3}\int_0^t \frac{1}{\sqrt{9-x^2}}\,dx = \lim_{t\to 3}(\arcsin(t/3) - \arcsin(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

(e) $\displaystyle\int_0^1 \ln x\,dx$
Vertical asymptote at $x = 0$
$= \lim_{t\to 0}\int_t^1 \ln x\,dx = \lim_{t\to 0}\left[x(\ln x - 1)\right]_t^1 = \lim_{t\to 0}(t - 1 - t\ln t) = 0 - 1 - 0 = -1$

Lecture Notes: Integrals with multiple improper pieces are a bit tricky to handle and most of us will want to find a short cut. But, here are the basic rules:

- Every improper piece must be handled separately.
- If one improper piece diverges then the whole integral diverges.

Here's a partial exam of what I mean by handling each piece separately. The integral below has vertical asymptotes at $x = 2$, $x = 4$, $x = 6$.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x^5+2x^2+1}{(x-2)(x-4)(x-6)(x^4+1)}\,dx = &\int_{-\infty}^{0} \frac{x^5+2x^2+1}{(x-2)(x-4)(x-6)(x^4+1)}\,dx \\ &+\int_{0}^{2} \frac{x^5+2x^2+1}{(x-2)(x-4)(x-6)(x^4+1)}\,dx \\ &+\int_{2}^{3} \frac{x^5+2x^2+1}{(x-2)(x-4)(x-6)(x^4+1)}\,dx \\ &+\int_{3}^{4} \frac{x^5+2x^2+1}{(x-2)(x-4)(x-6)(x^4+1)}\,dx \\ &+\int_{4}^{5} \frac{x^5+2x^2+1}{(x-2)(x-4)(x-6)(x^4+1)}\,dx \\ &+\int_{5}^{6} \frac{x^5+2x^2+1}{(x-2)(x-4)(x-6)(x^4+1)}\,dx \\ &+\int_{6}^{7} \frac{x^5+2x^2+1}{(x-2)(x-4)(x-6)(x^4+1)}\,dx \\ &+\int_{7}^{\infty} \frac{x^5+2x^2+1}{(x-2)(x-4)(x-6)(x^4+1)}\,dx\end{aligned}$$

Notice how I broke it up at each vertical asymptote twice–one limit on each side of the asymptote. And, we still have to write down the limits for each of these integrals!

NO, there isn't a methodical short cut (except for the next question).

Will this integral converge? If it diverges at one of the integrals, then it diverges at all of them.

6. For the following improper integrals, determine an easier function to compare the integrand to, make the comparison, then determine convergence or divergence.

   (a) $\displaystyle\int_2^{\infty} \frac{\cos^2 x}{x^2}\,dx$

   Solution: For all x we have $\frac{\cos^2 x}{x^2} < \frac{1}{x^2}$ and therefore $\int_2^{\infty} \frac{\cos^2 x}{x^2}\,dx < \int_2^{\infty} \frac{1}{x^2}\,dx$ so we have convergence.

   (b) $\displaystyle\int_3^{\infty} \frac{1}{x+e^x}\,dx$

   Solution: Note, you could try to compare: $\frac{1}{x+e^x} < \frac{1}{x}$, but $\int_3^{\infty} \frac{1}{x}\,dx$ diverges so this comparison isn't going to give you any information.
   So we compare $\frac{1}{x+e^x} < \frac{1}{e^x}$ for $x > 0$. Thus the integral converges.

   (c) $\displaystyle\int_2^{\infty} \frac{1}{\ln x}\,dx$

   Solution: Compare: $\frac{1}{\ln x} > \frac{1}{x}$, thus we have divergence. Note, this comparison is only valid for $x > e$.

   (d) $\displaystyle\int_2^{\infty} \frac{1}{\ln x - 1}\,dx$

   Solution: Compare: $\frac{1}{\ln x - 1} > \frac{1}{\ln x} > \frac{1}{x}$, thus we have divergence. Note, this comparison is only valid for $x > e$.

   (e) $\displaystyle\int_5^{\infty} \frac{x^2+x+1}{x^4+x^2+3}\,dx$

   Solution: We would like to compare to $\frac{1}{x^2}$. It is not true that $\frac{x^2+x+1}{x^4+x^2+3} < \frac{1}{x^2}$. But, for $x > 1$ we do have $x^2+x+1 < x^2+x^2+x^2 = 3x^2$. Thus $\frac{x^2+x+1}{x^4+x^2+3} < \frac{3x^2}{x^4+x^2+3} < \frac{3x^2}{x^4} = \frac{3}{x^2}$ and we have convergence.

   (f) $\displaystyle\int_5^{\infty} \frac{x^2+x+1}{x^3+x^2+3}\,dx$

   Solution: We would like to compare to $\frac{1}{x}$. It is not true that $\frac{x^2+x+1}{x^3+x^2+3} > \frac{1}{x}$. But, we do have, for $x > 1$: $\frac{x^2+x+1}{x^3+x^2+3} > \frac{x^2}{x^3+x^2+3} > \frac{x^2}{x^3+x^3+x^3} = \frac{1}{3x}$ and we have divergence.

7. Determine convergence or divergence

(a) $\displaystyle\int_5^8 \frac{6}{\sqrt{t-5}}\,dt$ **Solution:** Converges. Use $u = t - 5$ and then compare to $\int_0^3 \frac{1}{\sqrt{x}}\,dx$

(b) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^3+x}}$ **Solution:** Converges. Compare: $\frac{1}{\sqrt{x^3+x}} < \frac{1}{\sqrt{x}}$

(c) $\displaystyle\int_0^1 \frac{dx}{\sqrt{2x-x^2}}$ **Solution:** Converges. Compare: $\frac{1}{\sqrt{2x-x^2}} < \frac{1}{\sqrt{x}}$.
Why is this valid? For x in $(0, 1]$ we have $x^2 < x$ and $x < 2x - x^2$. Thus, $\sqrt{x} < \sqrt{2x - x^2}$ and $\frac{1}{\sqrt{2x-x^2}} < \frac{1}{\sqrt{x}}$

(d) $\displaystyle\int_0^\infty \frac{\sin 3x}{x^2+1}\,dx$ **Solution:** Converges. Compare: $\frac{\sin 3x}{x^2+1} < \frac{1}{x^2+1} < \frac{1}{x^2}$.

(e) $\displaystyle\int_1^\infty \frac{2}{x^{11}+x^{10}}\,dx$ **Solution:** Converges. Compare: $\frac{2}{x^{10}+x^{11}} < \frac{2}{x^{10}}$

(f) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+x}}\,dx$ **Solution:** Diverges. Compare $\frac{1}{\sqrt{x^2+x}} > \frac{1}{\sqrt{x^2+x^2}} = \frac{1}{\sqrt{2}\,x}$