Math 132: Discussion Session: Week 9

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–3	0%
4–6	80%
7–12	100%

Group Members:

7.3: Trigonometric Substitution.

(1) $\displaystyle\int \frac{dx}{(9-x^2)^{3/2}} =$

(2) $\displaystyle\int \frac{dx}{x\sqrt{x^2+25}} =$

(3) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+10}} =$

(4) $\displaystyle\int \frac{x^2}{\sqrt{5x^2-49}}\, dx =$

(5) $\displaystyle\int \frac{dx}{(x^2+a^2)^2} =$ for $a > 0$

7.4: The Method of Partial Fractions.

(1) $\displaystyle\int \frac{x^3+1}{x^2+1}\, dx =$

(2) $\displaystyle\int \frac{1}{(x+2)(x^2+4x+14)}\, dx =$

(3) $\displaystyle\int \frac{4x^2-12}{(2x+5)^3}\, dx =$

(4) $\displaystyle\int \frac{x^5}{x^4-1}\, dx =.$

(5) $\displaystyle\int \frac{10}{(x-1)^2(x^2+4)}\, dx =$

7.5: Strategy for Integration.

(1) $\displaystyle\int x^3(\ln x)^2\, dx =$

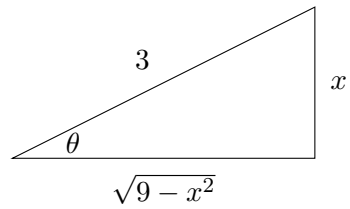
(2) $\displaystyle\int (5\sec x - \cos x)^2\, dx =$

Math 132 Discussion Session: Week 9

7.3: Trigonometric Substitution. Compute the following integrals using trigonometric subsitution. You might need to look up some trig integrals in the textbook.

(1) $\displaystyle\int \frac{dx}{(9-x^2)^{3/2}}$

Solution: We set up a triangle to get $\sqrt{9-x^2}$. We have



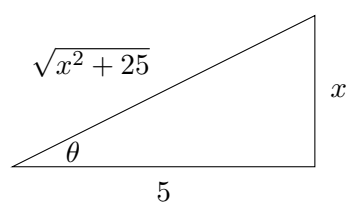
From the triangle, we see that $x = 3\sin\theta$ and $\sqrt{9-x^2} = 3\cos\theta$. Then $dx = 3\cos\theta\, d\theta$ and we can compute

$$\begin{aligned}\int \frac{dx}{(9-x^2)^{3/2}} &= \int \frac{3\cos\theta\, d\theta}{(3\cos\theta)^3}\, d\theta \\ &= \frac{1}{9}\int \sec^2\theta\, d\theta \\ &= \frac{1}{9}\tan\theta + C \\ &= \boxed{\frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C.}\end{aligned}$$

We used the triangle to find $\tan\theta$ in terms of x.

(2) $\displaystyle\int \frac{dx}{x\sqrt{x^2+25}}$

Solution: To get $\sqrt{x^2+25}$, we use the following triangle.



From the triangle, we see that $x = 5\tan\theta$. Thus $dx = 5\sec^2\theta\, d\theta$. The triangle also tells us that $\sqrt{x^2+25} = 5\sec\theta$, at which point we are ready to substitute.

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2+25}} &= \int \frac{5\sec^2\theta\, d\theta}{(5\tan\theta)(5\sec\theta)} \\ &= \frac{1}{5}\int \frac{\sec\theta}{\tan\theta}\, d\theta \\ &= \frac{1}{5}\int \csc\theta\, d\theta \\ &= \frac{1}{5}\ln|\csc\theta - \cot\theta| + C \\ &= \boxed{\frac{1}{5}\ln\left|\frac{\sqrt{x^2+25}}{x} - \frac{5}{x}\right| + C.}\end{aligned}$$

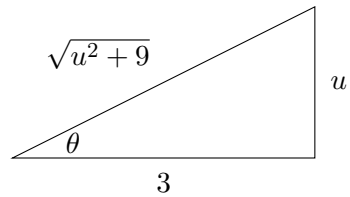
We needed to use a table of integrals to look up $\int \csc\theta\, d\theta$, and we used the triangle to write $\csc\theta$ and $\cot\theta$ in terms of x.

(3) $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+10}}$

Solution: The first step is to complete the square, getting $x^2+2x+10=(x+1)^2+9$. The next step is to substitute $u=x+1$, $du=dx$. Thus,

$$\int \frac{dx}{\sqrt{x^2+2x+10}} = \int \frac{du}{\sqrt{u^2+9}}.$$

We then set up a right triangle that has $\sqrt{u^2+9}$ as one of its lengths.



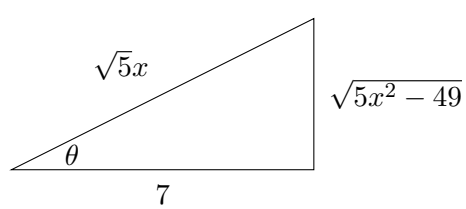
From the triangle, we can find that $u=3\tan\theta$, and $\sqrt{u^2+9}=3\sec\theta$. We can compute that $du = 3\sec^2\theta\,d\theta$. Then

$$\begin{aligned}\int \frac{du}{\sqrt{u^2+9}} &= \int \frac{3\sec^2\theta\,d\theta}{3\sec\theta}\\ &= \int \sec\theta\,d\theta\\ &= \ln|\sec\theta+\tan\theta|+C\\ &= \ln\left|\frac{\sqrt{u^2+9}}{3}+\frac{u}{3}\right|+C\\ &= \boxed{\ln\left|\frac{\sqrt{x^2+2x+10}}{3}+\frac{x+1}{3}\right|+C.}\end{aligned}$$

Note that using the substitution $u=3\sinh t$, we would obtain an equivalent answer of $\boxed{\sinh^{-1}\left(\frac{x+1}{3}\right)+C}$.

(4) $\displaystyle\int \frac{x^2}{\sqrt{5x^2-49}}\,dx$

Solution: One way to start would be to factor out a $\frac{1}{\sqrt{5}}$ to put the integral into a more familiar form, getting $\frac{1}{\sqrt{5}}\int \frac{x^2}{\sqrt{x^2-49/5}}\,dx$, getting us the substitution $x=\frac{7}{\sqrt{5}}\sec\theta$. We can also set up a triangle directly from the given integral. We want $\sqrt{5x^2-49}$ to appear as one of the lengths in the triangle, so we draw



As before, the triangle gives us $\sec\theta=\frac{\sqrt{5}x}{7}$, so $x=\frac{7}{\sqrt{5}}\sec\theta$, and so $dx=\frac{7}{\sqrt{5}}\tan\theta\sec\theta\,d\theta$. We also see from the triangle that $\sqrt{5x^2-49}=7\tan\theta$. We can then substitute and compute.

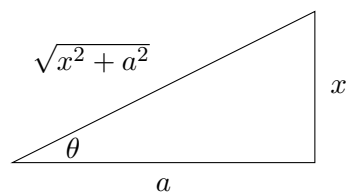
$$\begin{aligned}\int \frac{x^2}{\sqrt{5x^2-49}}\,dx &= \int \frac{\frac{49}{5}\sec^2\theta}{7\tan\theta}\frac{7}{\sqrt{5}}\tan\theta\sec\theta\,d\theta\\ &= \frac{49}{5\sqrt{5}}\int \sec^3\theta\,d\theta.\end{aligned}$$

We integrated $\sec^3\theta$ on a previous worksheet, and it's also one of the examples in section 7.2. We find that

$$\begin{aligned}
\frac{49}{5\sqrt{5}}\int \sec^3\theta\,d\theta &= \frac{49}{5\sqrt{5}}\cdot\frac{1}{2}\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)+C\\
&= \boxed{\frac{49}{10\sqrt{5}}\left(\frac{\sqrt{5}x}{7}\frac{\sqrt{5x^2-49}}{7}+\ln\left|\frac{\sqrt{5}x}{7}+\frac{\sqrt{5x^2-49}}{7}\right|\right)+C}\\
&= \boxed{\frac{1}{10}x\sqrt{5x^2-49}+\frac{49}{10\sqrt{5}}\ln\left|\sqrt{5}x+\sqrt{5x^2-49}\right|+D.}
\end{aligned}$$

(5) $\displaystyle\int \frac{dx}{(x^2+a^2)^2}$ for $a>0$

Solution: We draw a triangle to get $\sqrt{x^2+a^2}$.



From the triangle, we see that $x=a\tan\theta$, so $dx=a\sec^2\theta\,d\theta$. We also see from the triangle that $\sqrt{x^2+a^2}=a\sec\theta$, so $x^2+a^2=a^2\sec^2\theta$, and so $\left(x^2+a^2\right)^2=a^4\sec^4\theta$. Thus,

$$\begin{aligned}
\int\frac{dx}{(x^2+a^2)^2} &= \int\frac{a\sec^2\theta\,d\theta}{a^4\sec^4\theta}\\
&= \frac{1}{a^3}\int\cos^2\theta\,d\theta\\
&= \frac{1}{a^3}\int\frac{1}{2}\left(1+\cos 2\theta\right)\,d\theta\\
&= \frac{1}{2a^3}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)+C.
\end{aligned}$$

We have $\sin 2\theta$, but our triangle only tells us about trigonometric functions of θ, not 2θ. Thus, as in class, we need to use the formula $\sin 2\theta = 2\sin\theta\cos\theta$ to convert back to trigonometric functions of θ, so we can use the triangle to write them in terms of x. We have

$$\begin{aligned}
\frac{1}{2a^3}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)+C &= \frac{1}{2a^3}\left(\theta+\sin\theta\cos\theta\right)+C\\
&= \frac{1}{2a^3}\left(\arctan\left(\frac{x}{a}\right)+\frac{x}{\sqrt{x^2+a^2}}\cdot\frac{a}{\sqrt{x^2+a^2}}\right)+C\\
&= \boxed{\frac{1}{2a^3}\left(\arctan\left(\frac{x}{a}\right)+\frac{ax}{x^2+a^2}\right)+C.}
\end{aligned}$$

7.4: The Method of Partial Fractions. Compute the following integrals using the method of partial fractions:

(1) $\displaystyle\int\frac{x^3+1}{x^2+1}\,dx$

Solution: The degree of the numerator is at least the degree of the denominator, so we begin with long division.

$$
\begin{array}{rrrr|rrrr}
 & & & x & & & & \\
\cline{5-8}
x^2 & +0x & +1 & & x^3 & +0x^2 & +0x & +1 \\
 & & & & x^3 & & +x & \\
\cline{5-8}
 & & & & & & -x & +1
\end{array}
$$

Thus,

$$\frac{x^3+1}{x^2+1} = x + \frac{-x+1}{x^2+1} = x - \frac{x}{x^2+1} + \frac{1}{x^2+1}.$$

This expression is already in the partial fraction decomposition form, so we can just integrate it, using $u = x^2+1$ for the second term and $x = \tan\theta$ for the third term.

$$\int \frac{x^3+1}{x^2+1}\,dx = \boxed{\tfrac{1}{2}x^2 - \tfrac{1}{2}\ln(x^2+1) + \arctan x + C.}$$

(2) $\displaystyle\int \frac{1}{(x+2)(x^2+4x+14)}\,dx$

Solution: We can check that the second factor is irreducible by computing the discriminant b^2-4ac. The partial fraction decomposition form for this expression is

$$\frac{1}{(x+2)(x^4+4x+14)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4x+14}.$$

Clearing denominators, we find that

$$1 = A(x^2+4x+14) + (Bx+C)(x+2).$$

Plugging in $x=-2$ to make the second term zero, we can find that $1 = 10A$, so $A = \frac{1}{10}$. Plugging in $x=0$, we can find that $1 = 14A + 2C$. Since we know A, we can solve for $C = -\frac{1}{5}$. Finally, plugging in $x=1$, we find that

$$1 = A(19) + (B+C)(3) = 19A + 3B + 3C.$$

Since we know A and C, we can find that $B = -\frac{1}{10}$. Thus,

$$\frac{1}{(x+2)(x^4+4x+14)} = \frac{1}{10}\left(\frac{1}{x+2} - \frac{x+2}{x^2+4x+14}\right).$$

The first term is easy to integrate, but the second requires more work. We can complete the square to find that $x^2+4x+14 = (x+2)^2+10$. Thus, we can solve this integral with the substitution $u = x+2$, $du = dx$. We compute that

$$\begin{aligned}
\int \frac{1}{(x+2)(x^4+4x+14)} &= \frac{1}{10}\left(\int \frac{1}{x+2}\,dx - \int \frac{u}{u^2+10}\,du\right) \\
&= \tfrac{1}{10}\left(\ln|x+2| - \tfrac{1}{2}\ln(u^2+10)\right) + C \\
&= \boxed{\tfrac{1}{10}\left(\ln|x+2| - \tfrac{1}{2}\ln(x^2+4x+14)\right) + C.}
\end{aligned}$$

To compute the second integral, we used the substitution $v = u^2+10$. Since we did two substitutions, that means that we could have done the substitution $v = x^2+4x+14$ from the beginning, but it would have been hard to foresee that.

(3) $\displaystyle\int \frac{4x^2-12}{(2x+5)^3}\,dx$

Solution: The degree of the numerator is smaller than the degree of the denominator, so we proceed straight to the partial fraction decomposition form of

$$\frac{4x^2-12}{(2x+5)^3} = \frac{A}{2x+5} + \frac{B}{(2x+5)^2} + \frac{C}{(2x+5)^3}.$$

Clearing denominators, we have

$$\begin{aligned}4x^2 - 12 &= A(2x+5)^2 + B(2x+5) + C \\ &= A(4x^2 + 20x + 25) + B(2x+5) + C \\ &= 4Ax^2 + (20A + 2B)x + (25A + 5B + C).\end{aligned}$$

Equating coefficients, our system of equations is

$$\begin{aligned}4 &= 4A \\ 0 &= 20A + 2B \\ -12 &= 25A + 5B + C.\end{aligned}$$

This system is easy to solve, because the first equation just gives us $A = 1$, which we can plug into the second equation to find that $B = -10$, and we can plug both of these into the final equation to find that $-12 = 25 - 50 + C$, so $C = 13$. Thus,

$$\frac{4x^2 - 12}{(2x+5)^3} = \frac{1}{2x+5} - \frac{10}{(2x+5)^2} + \frac{13}{(2x+5)^3}.$$

We can integrate this expression making use of the substitution $u = 2x + 5$, $du = 2dx$. We get

$$\int \frac{4x^2 - 12}{(2x+5)^3}\,dx = \boxed{\tfrac{1}{2}\ln|2x+5| + \frac{5}{2x+5} - \frac{13}{4}\cdot\frac{1}{(2x+5)^2} + C.}$$

(4) $\displaystyle\int \frac{x^5}{x^4 - 1}\,dx.$

Solution: We begin with long division

$$\begin{array}{rrrrr|rrrrrr}
 & & & & & x & & & & & \\
\cline{7-12}
x^4 & +0x^3 & +0x^2 & +0x & -1 & x^5 & +0x^4 & +0x^3 & +0x^2 & +0x & +0 \\
 & & & & & x^5 & & & & -x & \\
\cline{6-12}
 & & & & & & & & & x & +0
\end{array}$$

We find that

$$\frac{x^5}{x^4 - 1} = x + \frac{x}{x^4 - 1}.$$

Next, we focus on the remainder term $\frac{x}{x^4-1}$ and factor the denominator, gettting $x^4 - 1 = (x^2+1)(x+1)(x-1)$. We can then set up the form of the partial fraction decomposition.

$$\frac{x}{x^4 - 1} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x+1} + \frac{D}{x-1}.$$

Clearing denominators, we find that

$$x = (Ax + B)(x+1)(x-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1).$$

Plugging in $x = -1$ so that most of the terms are zero, we can find that $-1 = -4C$, so $C = \frac{1}{4}$. Plugging in $x = 1$ so that most of the terms are zero, we can find that $1 = 4D$, so $D = \frac{1}{4}$. Plugging in $x = 0$, we can find that $0 = -B - C + D$. Since we know C and D, we can find that $B = 0$. Finally, plugging in $x = 2$, we find that

$$2 = (2A + B)(3)(1) + C(5)(1) + D(5)(3) = 6A + 3B + 5C + 15D = 6A + 5$$

We can compute that $A = -\frac{1}{2}$. Thus, we have

$$\frac{x^5}{x^4 - 1} = x - \frac{1}{2}\cdot\frac{x}{x^2+1} + \frac{1}{4}\cdot\frac{1}{x+1} + \frac{1}{4}\cdot\frac{1}{x-1}.$$

Integrating using $u = x^2 + 1$ for $\frac{x}{x^2+1}$ term, we find that

$$\int \frac{x^5}{x^4 - 1}\, dx = \boxed{\tfrac{1}{2}x^2 - \tfrac{1}{4}\ln(x^2+1) + \tfrac{1}{4}\ln|x+1| + \tfrac{1}{4}\ln|x-1| + C} = \boxed{\tfrac{1}{2}x^2 - \tfrac{1}{4}\ln\left|\frac{x^2-1}{x^2+1}\right|}.$$

(5) $\displaystyle\int \frac{10}{(x-1)^2(x^2+4)}\, dx$

Solution: The form of the partial fraction decomposition is

$$\frac{10}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4}$$

Clearing denominators, we find that

$$10 = A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2.$$

Plugging in $x = 1$ to make most of the terms zero, we can find that $10 = 5B$, so $B = 2$. Expanding, we find that

$$\begin{aligned} 10 &= A(x^3 - x^2 + 4x - 4) + 2(x^2+4) + (Cx^3 - 2Cx^2 + Cx + Dx^2 - 2Dx + D) \\ -2x^2 + 2 &= (A+C)x^3 + (-A - 2C + D)x^2 + (4A + C - 2D)x + (-4A + D). \end{aligned}$$

Grouping terms, we obtain the system of equations

$$\begin{aligned} 0 &= A + C \\ -2 &= -A - 2C + D \\ 0 &= 4A + C - 2D \\ 2 &= -4A + D. \end{aligned}$$

The first equation gives us that $C = -A$, so the remaining equations simplify as

$$\begin{aligned} -2 &= A + D \\ 0 &= 3A - 2D \\ 2 &= -4A + D. \end{aligned}$$

Subtracting the first and third equations, we find that $-4 = 5A$, so $A = -\frac{4}{5}$. Then, the first equation tells us that $D = -\frac{6}{5}$. We check that the second equation is also satisfied with this choice of A and D. And, we know that $C = -A = \frac{4}{5}$. Thus,

$$\frac{10}{(x-1)^2(x^2+4)} = -\frac{4}{5}\cdot\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{4}{5}\cdot\frac{x}{x^2+4} - \frac{6}{5}\cdot\frac{1}{x^2+4}$$

Integrating, using $u = x^2 + 4$ for the third term and $x = 2\tan\theta$ for the fourth term, we find that

$$\int \frac{10}{(x-1)^2(x^2+4)}\, dx = \boxed{-\tfrac{4}{5}\ln|x-1| - \frac{2}{x-1} + \tfrac{2}{5}\ln(x^2+4) - \tfrac{3}{5}\arctan\left(\tfrac{x}{2}\right) + C.}$$

7.5: Strategy for Integration. Compute the following integrals using any integration method you can.

(1) $\displaystyle\int x^3(\ln x)^2\, dx$

Solution: This integral is a product, so integration by parts is a reasonable thing to try. (Substitution $u = \ln x$ is also a reasonable thing to try, but if you try it, you'll find that the result is about the same complexity.) We can't easily integrate $(\ln x)^2$, but we can certainly differentiate that term using the chain rule. Thus, we try setting up an integration by parts as

$$\begin{aligned} f'(x) &= x^3, & g(x) &= (\ln x)^2, \\ f(x) &= \tfrac{1}{4}x^4, & g'(x) &= 2\ln x \cdot \tfrac{1}{x}. \end{aligned}$$

Thus,

$$\int x^3(\ln x)^2\, dx = \tfrac{1}{4}x^4(\ln x)^2 - \tfrac{1}{2}\int x^3 \ln x\, dx.$$

At this point, we take stock of the situation. Have we made progress? Yes, the new integral $\int x^3 \ln x \, dx$ is simpler than the original problem $\int x^3 (\ln x)^2$. Our strategy seems promising, so we might try the same strategy again.

$$f'(x) = x^3, \qquad g(x) = \ln x$$
$$f(x) = \tfrac{1}{4}x^4, \qquad g'(x) = \tfrac{1}{x}.$$

Thus,

$$\begin{aligned}\int x^3 \ln x \, dx &= \tfrac{1}{4}x^4 \ln x - \tfrac{1}{4}\int x^3 \, dx \\ &= \tfrac{1}{4}x^4 \ln x - \tfrac{1}{16}x^4 + C.\end{aligned}$$

We've computed this integral, so now we just need to put all of our work together and write the answer.

$$\int x^3 (\ln x)^2 \, dx = \boxed{\tfrac{1}{4}x^4 (\ln x)^2 - \tfrac{1}{8}x^4 \ln x + \tfrac{1}{32}x^4 + D.}$$

(2) $\displaystyle\int (5 \sec x - \cos x)^2 \, dx$

Solution: We should clearly use some kind of trig integration techniques, but those are all about products of trig functions, which this expression isn't. But, of the first three things we might think of trying, expanding out the expression into three integrals should be among them. We see that

$$\begin{aligned}\int (5 \sec x - \cos x)^2 \, dx &= \int \left(25 \sec^2 x - 10 \sec x \cos x + \cos^2 x\right) \, dx \\ &= 25 \int \sec^2 x \, dx - \int 10 \, dx + \int \cos^2 \, dx.\end{aligned}$$

The first integral we've seen before: it's just $\tan x$. The second integral is easy, and the third integral we've also seen before: it needs a double angle formula. Working out the third integral, we find that

$$\int \cos^2 x \, dx = \frac{1}{2}\int (1 + \cos 2x) \, dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C.$$

Thus, the entire integral is

$$\begin{aligned}\int (5 \sec x - \cos x)^2 \, dx &= \tan x - 10x + \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C \\ &= \boxed{25 \tan x - \tfrac{19}{2}x + \tfrac{1}{4}\sin 2x + C.}\end{aligned}$$