Math 132 - October 13, 2017

Solutions

7.3 Trig Substitution	**7.4 Partial Fractions**
• Completing the square	• Polynomial Long Division

Warm-up Problems

1. What is a rational function?

   Solution: A quotient of polynomials. To integrate rational functions, we are going to use *Partial Fractions*, an algebraic technique that simplifies rational functions to something easier to integrate.

2. What are the trig identities used in trig subsitution?

   Solution:

   - $\sin^2\theta + \cos^2\theta = 1$
   - $\tan^2\theta + 1 = \sec^2\theta$

3. $\boxed{\textbf{Clicker}}$ Complete the square $x^2 - 6x + 13 =???$

   (a) $x^2 - 6x + 13 = (x+3)^2 + 4$

   (b) $x^2 - 6x + 13 = (x-3)^2 + 4$ $\boxed{\textbf{Correct}}$

   (c) $x^2 - 6x + 13 = (x-3)^2 + 13$

   (d) $x^2 - 6x + 13 = (x+3)^2 + 13$

> **Lecture Notes:** The point here is that completing the square can help you integrate. See Problem 4 to see how you might integrate with this complete square.

Class Problems

4. $\displaystyle\int_3^5 \frac{1}{\sqrt{x^2 - 6x + 13}}\,dx$

   Hint: Use the completed square from Problem 3.

   Solution: Complete the square $x^2 - 6x + 13 = (x-3)^2 + 4$ and make the substitution $u = x - 3$

$$
\begin{aligned}
&= \int \frac{1}{\sqrt{u^2+4}}\,dx \qquad \text{(Now use } u = 2\tan\theta\text{)}\\
&= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta\\
&= \ln|\sec\theta + \tan\theta|\Big|_{\theta=0}^{\theta=\pi/4}\\
&= (\ln|\sec(\pi/4) + \tan(\pi/4)|) - (\ln|\sec(0) + \tan(0)|)\\
&= \Big(\ln|\sqrt(2) + 1|\Big) - (\ln|1|) = \ln(\sqrt(2) + 1)
\end{aligned}
$$

   Note: For limits, $x = 3 \implies u = 0 \implies \theta = 0$ and $x = 5 \implies u = 2 \implies \theta = \pi/4$ and

5. $\displaystyle\int \sqrt{x^2 + 4x + 5}\,dx$

   Solution:

$$
\begin{aligned}
\int \sqrt{x^2+4x+5}\,dx = \int \sqrt{(x+2)^2+1}\,dx &= \int \sqrt{\tan^2\theta + 1}\,\sec^2\theta\,d\theta \qquad x = \tan\theta\\
&= \int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C\\
&= \frac{1}{2}(x+2)\sqrt{x^2+4x+5} + \frac{1}{2}\ln\left|x + 2 + \sqrt{x^2+4x+5}\right| + C
\end{aligned}
$$

6. $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$

Solution: Use $x = 2\sin\theta$

$$
\begin{aligned}
&= \int_0^{\pi/6} \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta \\
&= \int_0^{\pi/6} 4\cos^2\theta\,d\theta \\
&= \int_0^{\pi/6} 2(1+\cos 2\theta)\,d\theta \\
&= 2\theta + \sin(2\theta)\Big|_0^{\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}
\end{aligned}
$$

Note: if you wanted to convert back to x's, you would probably need to use $\sin 2\theta = 2\sin\theta\cos\theta$.

7. $\displaystyle\int_1^2 \frac{1}{x^2\sqrt{x^2+16}}\,dx$

Solution: use $x = 4\tan\theta$

$$
\begin{aligned}
&= \int \frac{4\sec^2\theta}{16\tan^2\theta\cdot\sqrt{16\tan^2\theta+16}}\,d\theta \\
&= \frac{1}{16}\int \frac{\sec^2\theta}{\tan^2\theta\cdot\sec\theta}\,d\theta \\
&= \frac{1}{16}\int \frac{\sec\theta}{\tan\theta}\,d\theta \\
&= \frac{1}{16}\int \csc\theta\cot\theta\,d\theta \\
&= -\frac{1}{16}\csc\theta\Big|_{x=1}^{x=2} \\
&= -\frac{1}{16}\frac{\sqrt{x^2+16}}{x}\Big|_{x=1}^{x=2} = \frac{\sqrt{17}-\sqrt{5}}{16}
\end{aligned}
$$

Note: In this case, it seemed easier to me to convert back to the x's.

8. $\displaystyle\int_0^4 x^2\sqrt{16-x^2}\,dx$

Solution: Use $x = 4\sin\theta$

$$
\begin{aligned}
&= \int_0^{\pi/2} 16\sin^2\theta\sqrt{16-16\sin^2\theta}\cdot 4\cos\theta\,d\theta \\
&= 256\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta \\
&= 256\int_0^{\pi/2} \frac{1}{4}(1-\cos 2\theta)(1+\cos 2\theta)\,d\theta \\
&= 64\int_0^{\pi/2} (1-\cos^2 2\theta)\,d\theta \\
&= 64\int_0^{\pi/2} \left[1-\frac{1}{2}(1+\cos 4\theta)\right]\,d\theta \\
&= 64\int_0^{\pi/2} \frac{1}{2}(1-\cos 4\theta)\,d\theta \\
&= 32\int_0^{\pi/2} (1-\cos 4\theta)\,d\theta \\
&= 32\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = 16\pi
\end{aligned}
$$

Lecture Notes: **Long Division and Partial Fractions** We need to be able to perform long division on polynomials to integrate rational functions. Long division with polynomials is exactly like long division of integers (which I have heard isn't taught as much as it should be prior to calculus).

Here's an easy example to compute $\frac{24384}{7}$ or

$$
\begin{array}{r}
3483 \\
7\overline{)24384} \\
\underline{21000} \\
3384 \\
\underline{2800} \\
584 \\
\underline{560} \\
24 \\
\underline{21} \\
3
\end{array}
$$

Sometimes people write (incorrectly, because its not really a meaningful equality!) that $24384 \div 7 = 3483\,R\,3$. Instead, you should use the long division work to conclude

$$\frac{24384}{7} = 3483 + \frac{3}{7}$$

Be sure to see the following:

1. Where everything in the equation above came from (how did the long division help create the equation?)
2. That the equation above is a true equation (both sides are equal)

We do exactly the same thing to perform polynomial long division. Here is $\dfrac{x^3+x^2-1}{x-1}$:

$$
\begin{array}{r}
x^2+2x+2 \\
x-1\overline{)\;x^3+x^2\qquad-1} \\
\underline{-x^3+x^2\qquad\quad} \\
2x^2\qquad\quad \\
\underline{-2x^2+2x\quad} \\
2x-1 \\
\underline{-2x+2} \\
1
\end{array}
$$

We write this as

$$\frac{x^3+x^2-1}{x-1} = x^2+2x+2+\frac{1}{x-1}$$

exactly like we would write in integer long division.

9. Perform polynomial long division and write your answers correctly.

(a) $\dfrac{x^2+1}{x+2}$

Solution: $\frac{x^2+1}{x+2} = x-2+\frac{5}{x+2}$

$$
\begin{array}{r}
x-2 \\
x+2\overline{)\;x^2\qquad+1} \\
\underline{-x^2-2x\quad} \\
-2x+1 \\
\underline{2x+4} \\
5
\end{array}
$$

(b) $\dfrac{x^2+1}{x^2+2}$

Solution: $\frac{x^2+1}{x^2+2} = 1 + \frac{-1}{x^2+2}$

$$
\begin{array}{r|l}
 & 1 \\ \cline{2-2}
x^2+2 & x^2+1 \\
 & -x^2-2 \\ \cline{2-2}
 & -1
\end{array}
$$

(c) $\dfrac{2x^3-x^2+5x+1}{3x+2}$

Solution: $\frac{2x^3-x^2+5x+1}{3x+2} = \frac{2}{3}x^2 - \frac{7}{9}x + \frac{59}{27} + \frac{-91/27}{3x+2}$

$$
\begin{array}{r|rrrr}
 & \frac{2}{3}x^2 & -\frac{7}{9}x & +\frac{59}{27} & \\ \cline{2-5}
3x+2 & 2x^3 & -x^2 & +5x & +1 \\
 & -2x^3 & -\frac{4}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{7}{3}x^2 & +5x & \\
 & & \frac{7}{3}x^2 & +\frac{14}{9}x & \\ \cline{3-4}
 & & & \frac{59}{9}x & +1 \\
 & & & -\frac{59}{9}x & -\frac{118}{27} \\ \cline{4-5}
 & & & & -\frac{91}{27}
\end{array}
$$

(d) $\dfrac{2x^5-x^2+5x+1}{(x-2)(x^2+1)}$

Solution: $\frac{2x^5-x^2+5x+1}{(x-2)(x^2+1)} = 2x^2+4x+6+\frac{11x^2+7x+13}{x^3-2x^2+x-2}$

$$
\begin{array}{r|rrrrr}
 & & & 2x^2 & +4x & +6 \\ \cline{2-6}
x^3-2x^2+x-2 & 2x^5 & & -x^2 & +5x & +1 \\
 & -2x^5 & +4x^4-2x^3 & +4x^2 & & \\ \cline{2-4}
 & & 4x^4-2x^3 & +3x^2 & +5x & \\
 & & -4x^4+8x^3 & -4x^2 & +8x & \\ \cline{3-5}
 & & & 6x^3 & -x^2+13x & +1 \\
 & & & -6x^3+12x^2 & -6x+12 & \\ \cline{4-6}
 & & & & 11x^2 & +7x+13
\end{array}
$$

10. Find the general integrals below. (These are the types of integrals that should appear when partial fractions is conducted correctly.)

(a) $\displaystyle\int \frac{1}{ax+b}\,dx$

Solution: Use substitution, $u = ax+b$:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + C$$

(b) $\displaystyle\int \frac{1}{(ax+b)^2}\,dx$

Solution: Use substitution $u = ax+b$:

$$\int \frac{1}{(ax+b)^2}\,dx = -\frac{1}{a(ax+b)} + C$$

Note: This is similar to any integral of the form $\frac{1}{(ax+b)^n}$ for any $n > 1$.

(c) $\displaystyle\int \frac{x}{x^2+a^2}\,dx$

Solution: Use substitution $u = x^2 + a^2$:

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + C$$

(d) $\displaystyle\int \frac{x}{(x^2+a^2)^2}\,dx$

Solution: Use substitution $u = x^2 + a^2$:

$$\int \frac{x}{(x^2+a^2)^2}\,dx = -\frac{1}{2(x^2+a^2)} + C$$

Note: This is similar to any integral of the form $\frac{x}{(x^2+a)^n}$ for any $n > 1$.

(e) $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

Solution: You can use substitution $x = au$:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

(f) $\displaystyle\int \frac{1}{(x^2+a^2)^2}\,dx$

Solution: Use Trig subsitution $x = a\tan\theta$:

$$\int \frac{1}{(x^2+a^2)^2}\,dx = \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)} + C$$

In this case, integrals of the form $\int \frac{1}{(x^2+a^2)^n}\,dx$ are clearly trickier. You can do these using trig substitution ($x = a\tan\theta$)

$$\begin{aligned}\int \frac{1}{(x^2+a^2)^n}\,dx &= \int \frac{1}{(a^2\tan^2\theta + a^2)^n}\,d\theta \qquad x = a\tan\theta \\ &= \frac{1}{a^{2n}}\int \frac{1}{(\tan^2\theta+1)^n}\,d\theta = \frac{1}{a^{2n}}\int \frac{1}{\sec^{2n}\theta}\,d\theta = \frac{1}{a^{2n}}\int \cos^{2n}\theta\,d\theta\end{aligned}$$