7.3 Trig Substitution

- Completing the square

7.4 Partial Fractions

- Polynomial Long Division

Warm-up Problems

1. What is a rational function?

Solution: A quotient of polynomials. To integrate rational functions, we are going to use Partial Fractions, an algebraic technique that simplifies rational functions to something easier to integrate.

2. What are the trig identities used in trig substitution?

Solution:

- $\sin^2 \theta + \cos^2 \theta = 1$
- $\tan^2 \theta + 1 = \sec^2 \theta$

3. Clicker Complete the square $x^2 - 6x + 13 = ??$

(a) $x^2 - 6x + 13 = (x + 3)^2 + 4$
(b) $x^2 - 6x + 13 = (x - 3)^2 + 4$ [Correct]
(c) $x^2 - 6x + 13 = (x - 3)^2 + 13$
(d) $x^2 - 6x + 13 = (x + 3)^2 + 13$

Lecture Notes: The point here is that completing the square can help you integrate. See Problem 4 to see how you might integrate with this complete square.

Class Problems

4. $\int_{3}^{5} \frac{1}{\sqrt{x^2 - 6x + 13}} \, dx$

Hint: Use the completed square from Problem 3

Solution: Complete the square $x^2 - 6x + 13 = (x - 3)^2 + 4$ and make the substitution $u = x - 3$

\[
\int \frac{1}{\sqrt{u^2 + 4}} \, du = \int \frac{2 \sec^2 \theta}{2 \sec \theta} \, d\theta = \ln |\sec \theta + \tan \theta| \bigg|_{\theta=0}^{\theta=\pi/4} = (\ln |\sec(\pi/4) + \tan(\pi/4)|) - (\ln |\sec(0) + \tan(0)|) = \left(\ln |\sqrt{2} + 1| \right) - (\ln |1|) = \ln(\sqrt{2} + 1)
\]

Note: For limits, $x = 3 \implies u = 0 \implies \theta = 0$ and $x = 5 \implies u = 2 \implies \theta = \pi/4$ and

5. $\int \sqrt{x^2 + 4x + 5} \, dx$

Solution:

\[
\int \sqrt{x^2 + 4x + 5} \, dx = \int \sqrt{(x + 2)^2 + 1} \, dx = \int \sqrt{\tan^2 \theta + 1} \, sec \theta \, d\theta \quad x = \tan \theta
\]

\[
= \int sec^3 \theta \, d\theta = \frac{1}{2} sec \theta \tan \theta + \frac{1}{2} \ln |sec \theta + tan \theta| + C
\]

\[
= \frac{1}{2} (x + 2) \sqrt{x^2 + 4x + 5} + \frac{1}{2} \ln \left| x + 2 + \sqrt{x^2 + 4x + 5} \right| + C
\]
6. \( \int_0^1 \sqrt{4 - x^2} \, dx \)

**Solution:** Use \( x = 2 \sin \theta \)

\[
= \int_0^{\pi/6} \sqrt{4 - 4 \sin^2 \theta} \cdot 2 \cos \theta \, d\theta \\
= \int_0^{\pi/6} 4 \cos^2 \theta \, d\theta \\
= \int_0^{\pi/6} 2(1 + \cos 2\theta) \, d\theta \\
= 2\theta + \sin(2\theta) \bigg|_0^{\pi/6} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}
\]

Note: if you wanted to convert back to \( x \)'s, you would probably need to use \( \sin 2\theta = 2 \sin \theta \cos \theta \).

7. \( \int_1^2 \frac{1}{x^2\sqrt{x^2 + 16}} \, dx \)

**Solution:** use \( x = 4 \tan \theta \)

\[
= \int_1^2 \frac{4 \sec^2 \theta}{16 \tan^2 \theta \cdot \sqrt{16 \tan^2 \theta + 16}} \, d\theta \\
= \frac{1}{16} \int_1^2 \frac{\sec^2 \theta \cdot \sec \theta}{\tan^2 \theta \cdot \sec \theta} \, d\theta \\
= \frac{1}{16} \int_1^2 \sec \theta \cdot \tan \theta \, d\theta \\
= \frac{1}{16} \csc \theta \bigg|_{x=1}^{x=2} \\
= -\frac{1}{16} \frac{\sqrt{x^2 + 16}}{x} \bigg|_{x=1}^{x=2} = \frac{\sqrt{17} - \sqrt{5}}{16}
\]

Note: In this case, it seemed easier to me to convert back to the \( x \)'s.

8. \( \int_0^4 x^2 \sqrt{16 - x^2} \, dx \)

**Solution:** Use \( x = 4 \sin \theta \)

\[
= \int_0^{\pi/2} 16 \sin^2 \theta \sqrt{16 - 16 \sin^2 \theta} \cdot 4 \cos \theta \, d\theta \\
= 256 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta \, d\theta \\
= 256 \int_0^{\pi/2} \frac{1}{4} (1 - \cos 2\theta)(1 + \cos 2\theta) \, d\theta \\
= 64 \int_0^{\pi/2} (1 - \cos^2 2\theta) \, d\theta \\
= 64 \int_0^{\pi/2} \left[ 1 - \frac{1}{2} \right] \, d\theta \\
= 64 \int_0^{\pi/2} \frac{1}{2} (1 - \cos 4\theta) \, d\theta \\
= 32 \int_0^{\pi/2} (1 - \cos 4\theta) \, d\theta \\
= 32 \left[ \theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/2} = 16\pi
\]
Lecture Notes: **Long Division and Partial Fractions** We need to be able to perform long division on polynomials to integrate rational functions. Long division with polynomials is exactly like long division of integers (which I have heard isn’t taught as much as it should be prior to calculus).

Here’s an easy example to compute $\frac{24384}{7}$ or

\[
\begin{array}{c|cccc}
& 3483 \\
7 & 24384 \\
\cline{2-5}
& 21000 \\
& 3384 \\
\cline{2-4}
& 2800 \\
& 584 \\
\cline{2-3}
& 560 \\
& 24 \\
\cline{2-2}
& 21 \\
\cline{2-1}
& 3 \\
\end{array}
\]

Sometimes people write (incorrectly, because its not really a meaningful equality!) that $24384 \div 7 = 3483 R 3$. Instead, you should use the long division work to conclude

\[
\frac{24384}{7} = 3483 + \frac{3}{7}
\]

Be sure to see the following:

1. Where everything in the equation above came from (how did the long division help create the equation?)
2. That the equation above is a true equation (both sides are equal)

We do exactly the same thing to perform polynomial long division. Here is $\frac{x^3 + x^2 - 1}{x - 1}$:

\[
\begin{array}{c}
x^2 + 2x + 2 \\
-x^3 + x^2 - 1 \\
\hline
2x^2 + 2x \\
-2x^3 + 2x \\
\hline
2x - 1 \\
-2x + 2 \\
\hline
1 \\
\end{array}
\]

We write this as

\[
\frac{x^3 + x^2 - 1}{x - 1} = x^2 + 2x + 2 + \frac{1}{x - 1}
\]

exactly like we would write in integer long division.

9. Perform polynomial long division and write your answers correctly.

(a) $\frac{x^2 + 1}{x + 2}$

**Solution:** $\frac{x^2 + 1}{x + 2} = x - 2 + \frac{5}{x + 2}$

\[
\begin{array}{c|cc}
x + 2 & x^2 & + 1 \\
\cline{2-3}
& -x^2 - 2x \\
\cline{2-2}
& -2x + 1 \\
\cline{2-1}
& 2x + 4 \\
\cline{2-1}
& 5 \\
\end{array}
\]
(b) \[ \frac{x^2 + 1}{x^2 + 2} \]

Solution: \[ \frac{x^2 + 1}{x^2 + 2} = 1 + \frac{-1}{x^2 + 2} \]

\[ x^2 + 2 \]

\[ \frac{1}{-x^2 - 2} \]

\[ -1 \]

(c) \[ \frac{2x^3 - x^2 + 5x + 1}{3x + 2} \]

Solution: \[ \frac{2x^3 - x^2 + 5x + 1}{3x + 2} = \frac{2x^3}{3} - \frac{x^2}{3} + \frac{5x}{3} + \frac{-91/27}{3x + 2} \]

\[ 3x + 2 \]

\[ \frac{2x^3}{3} - \frac{x^2}{3} + \frac{5x}{3} \]

\[ - \frac{7x^2}{3x + 2} + \frac{14x}{9x + 27} \]

\[ \frac{59x}{9x + 27} + 1 \]

\[ - \frac{59x}{9x + 27} - \frac{118}{27} \]

\[ - \frac{91}{27} \]

(d) \[ \frac{2x^5 - x^2 + 5x + 1}{(x - 2)(x^2 + 1)} \]

Solution: \[ \frac{2x^5 - x^2 + 5x + 1}{(x - 2)(x^2 + 1)} = 2x^2 + 4x + 6 + \frac{11x^2 + 7x + 13}{x^2 - 2x^2 + x - 2} \]

\[ x^3 - 2x^2 + x - 2 \]

\[ \frac{2x^5}{x^2 - 2x^2 + x - 2} - \frac{x^2 + 5x + 1}{x^2 - 2x^2 + x - 2} \]

\[ - \frac{2x^5 + 4x^4 - 2x^3 + 4x^2}{4x^4 - 2x^3 + 3x^2 + 5x} \]

\[ - \frac{4x^4 + 8x^3 - 4x^2 + 8x}{-4x^4 + 8x^3 - 4x^2 + 8x} \]

\[ \frac{6x^3}{6x^3 - 13x + 1} \]

\[ - \frac{6x^3 + 12x^2}{-6x + 1} \]

\[ - \frac{11x^2 + 7x + 13}{11x^2 + 7x + 13} \]

10. Find the general integrals below. (These are the types of integrals that should appear when partial fractions is conducted correctly.)

(a) \[ \int \frac{1}{ax + b} \, dx \]

Solution: Use substitution, \( u = ax + b \):

\[ \int \frac{1}{ax + b} \, dx = \frac{1}{a} \ln(ax + b) + C \]

(b) \[ \int \frac{1}{(ax + b)^2} \, dx \]

Solution: Use substitution \( u = ax + b \):

\[ \int \frac{1}{(ax + b)^2} \, dx = -\frac{1}{a(ax + b)} + C \]

Note: This is similar to any integral of the form \( \frac{1}{(ax + b)^n} \) for any \( n > 1 \).
(c) $\int \frac{x}{x^2 + a^2} \, dx$

**Solution:** Use substitution $u = x^2 + a^2$:

$$\int \frac{x}{x^2 + a^2} \, dx = \frac{1}{2} \ln(x^2 + a^2) + C$$

(d) $\int \frac{x}{(x^2 + a^2)^2} \, dx$

**Solution:** Use substitution $u = x^2 + a^2$:

$$\int \frac{x}{(x^2 + a^2)^2} \, dx = -\frac{1}{2(x^2 + a^2)} + C$$

Note: This is similar to any integral of the form $\frac{x}{(x^2 + a)^n}$ for any $n > 1$.

(e) $\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C$

**Solution:** You can use substitution $x = au$:

$$\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C$$

(f) $\int \frac{1}{(x^2 + a^2)^2} \, dx$

**Solution:** Use Trig substitution $x = a \tan \theta$:

$$\int \frac{1}{(x^2 + a^2)^2} \, dx = \frac{1}{2a^3} \arctan \left( \frac{x}{a} \right) + \frac{x}{2a^2(x^2 + a^2)} + C$$

In this case, integrals of the form $\int \frac{1}{(x^2 + a^2)^n} \, dx$ are clearly trickier. You can do these using trig substitution ($x = a \tan \theta$)

$$\int \frac{1}{(x^2 + a^2)^n} \, dx = \int \frac{1}{(a^2 \tan^2 \theta + a^2)^n} \, d\theta \quad x = a \tan \theta \quad \Rightarrow \quad d\theta = \frac{1}{a^2n} \int \frac{1}{(\tan^2 \theta + 1)^n} \, d\theta = \frac{1}{a^2n} \int \frac{1}{\sec^2 \theta} \, d\theta = \frac{1}{a^2n} \int \cos^{2n} \theta \, d\theta$$