Math 132 - October 11, 2017

Solutions

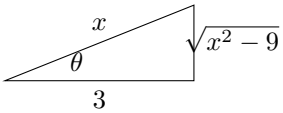
Sections Covered: 7.3 Trig Substitution

1. Determine the correct trig substitution.
2. Substitute (don't forget dx, $d\theta$)
3. Simplify and integrate.
4. Definite Integral: change limits.
5. Indefinite: draw the trig subsitution triangle. Get x's back.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Warm-up Problems

1. Given the triangle below, find the values of the trig functions asked for.



   (a) $\sin\theta$
   Solution: $\sin\theta = (\sqrt{x^2-9})/x$
   (b) $\cos\theta$
   Solution: $\cos\theta = 3/x$
   (c) $\tan\theta$
   Solution: $\tan\theta = (\sqrt{x^2-9})/3$
   (d) $\csc\theta$
   Solution: $\csc\theta = x/\sqrt{x^2-9}$
   (e) $\sec\theta$
   Solution: $\sec\theta = x/3$
   (f) $\cot\theta$
   Solution: $\cot\theta = 3/\sqrt{x^2-9}$

2. **Clicker** Starting with the expression below you are to make a substitution that will eliminate the square root. Choose the best substitution.

$$x\sqrt{1-x^2}$$

   (a) $x = \sin\theta$ **Correct** (b) $x = \cos\theta$ **Correct** (c) $x = \tan\theta$ (d) $x = \sec\theta$
   Solution: The substitition $x = \sin\theta$ gives

$$x\sqrt{1-x^2} = \sin\theta\sqrt{1-\sin^2\theta} = \sin\theta\sqrt{\cos^2\theta} = \sin\theta\cos\theta$$

Lecture Notes: Since we can use either one, we will usually use $x = \sin\theta$. There are a few reasons that $\sin\theta$ is slightly better than $\cos\theta$, but for most situations it is a bit arbitrary.

3. **Clicker** Starting with the expression below you are to make a substitution that will eliminate the square root. Choose the best substitution.

$$\frac{x^2}{\sqrt{x^2-1}}$$

   (a) $x = \sin\theta$ (b) $x = \cos\theta$ (c) $x = \tan\theta$ (d) $x = \sec\theta$ **Correct**
   Solution: The substitution $x = \tan\theta$ gives

$$\frac{x^2}{\sqrt{x^2-1}} = \frac{\sec^2\theta}{\sqrt{\sec^2\theta - 1}} = \frac{\sec^2\theta}{\sqrt{\tan^2\theta}} = \frac{\sec^2\theta}{\tan\theta}$$

4. $\boxed{\textbf{Clicker}}$ Starting with the expression below you are to make a substitution that will eliminate the square root. Choose the best substitution.

$$\frac{\sqrt{1+x^2}}{x^2}$$

(a) $x = \sin\theta$ (b) $x = \cos\theta$ (c) $x = \tan\theta$ $\boxed{\textbf{Correct}}$ (d) $x = \sec\theta$

Solution: The substitution $x = \tan\theta$ gives

$$\frac{\sqrt{1+x^2}}{x^2} = \frac{\sqrt{1+\tan^2\theta}}{\tan^2\theta} = \frac{\sqrt{\sec^2\theta}}{\tan^2\theta} = \frac{\sec\theta}{\tan^2\theta}$$

Class Problems

Lecture Notes: The idea of trig substitution is to exploit the properties of trig functions to eliminate square roots in integrals. This is just like the previous problems–make the substitution required and make sure you take care of the dx and $d\theta$ properly.
Steps for trig substituion

I. Determine the correct substitution based on the trig identities.

II. Make the substitution, don't forget the dx and $d\theta$.

III. Simplify the integral as needed and integrate.

IV. If a definite integral, change the x-limits to θ-limits (this is usually a bit easier but not required).

V. If indefinite integral, draw the trig subsitution triangle. Using your triangle, resubstitute x's for the θ's making sure you have nothing that looks like $\sin\left(\cos^{-1} x\right)$.

Expression	**Substitution**	**Identity**
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

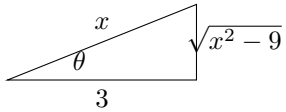
Lecture Notes: Example.

$$\begin{aligned}
\int_3^6 \frac{\sqrt{x^2-9}}{x}\,dx &= \int_0^{\pi/3} \frac{3\tan\theta}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta \qquad x = 3\sec\theta, dx = 3\sec\theta\tan\theta \\
&= 3\int_0^{\pi/3} \tan^2\theta\,d\theta \\
&= 3\int_0^{\pi/3} (\sec^2\theta - 1)\,d\theta \\
&= 3(\tan\theta - \theta)\Big|_0^{\pi/3} = 3\sqrt{3} - \pi
\end{aligned}$$

If this had been an indefinite integral its more difficult:

$$\begin{aligned}
\int \frac{\sqrt{x^2-9}}{x}\,dx &= \int \frac{3\tan\theta}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta \qquad x = 3\sec\theta, dx = 3\sec\theta\tan\theta \\
&= 3\int \tan^2\theta\,d\theta \\
&= 3\int (\sec^2\theta - 1)\,d\theta \\
&= 3(\tan\theta - \theta)
\end{aligned}$$

At this point we draw a "trig sub triangle" for the substitution $\sec\theta = \frac{x}{3}$



When you draw your triangle, you should see the square root that you are trying to eliminate.
From the triangle, you can read off trig functions of your angle. This avoids finishing the problem as $\theta = \cos^{-1}(3/x)$:

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = = 3\tan\theta - 3\theta + C = 3\tan\left(\cos^{-1}(3/x)\right) - 3\cos^{-1}(3/x) + C$$

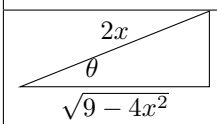
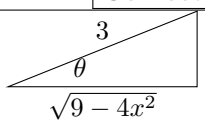
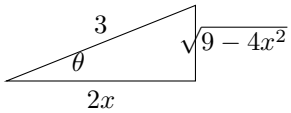
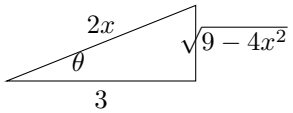
What is better is to read $\tan\theta$ off the triangle. This improves one of the terms and the other term just has to stay with $\cos^{-1}$.

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = = 3\tan\theta - 3\theta + C = 3\cdot\frac{\sqrt{x^2-9}}{3} - 3\cos^{-1}(3/x) + C$$

5. Integrate $\displaystyle\int \frac{\sqrt{9-4x^2}}{x}\,dx$

(a) **Clicker** Using the chart, determine the best substitution
A. $x = 3\sin\theta$ B. $x = \frac{3}{2}\sin\theta$ **Correct** C. $x = 3\sec\theta$ D. $x = \frac{3}{2}\sec\theta$ E. $x = 3\tan\theta$

(b) **Clicker** Draw the triangle associated to your substitution.

A.	**B.** **Correct**	**C.**	**D.**
			

(c) Using your triangle, find the following.

i. $\sin\theta$
Solution: $\sin\theta = \frac{2x}{3}$

ii. $\cos\theta$
Solution: $\cos\theta = \frac{\sqrt{9-4x^2}}{3}$

iii. $\tan\theta$
Solution: $\tan\theta = \frac{2x}{\sqrt{9-4x^2}}$

iv. $\csc\theta$
Solution: $\csc\theta = \frac{3}{2x}$

v. $\sec\theta$
Solution: $\sec\theta = \frac{3}{\sqrt{9-4x^2}}$

vi. $\cot\theta$
Solution: $\cot\theta = \frac{\sqrt{9-4x^2}}{2x}$

(d) Compute the integral

Solution:

$$\begin{aligned}
\int \frac{\sqrt{9-4x^2}}{x}\,dx &= \int \frac{\sqrt{9-9\sin^2\theta}}{\frac{3}{2}\sin\theta}\,\frac{3}{2}\cos\theta\,d\theta \qquad x=\frac{3}{2}\sin\theta, \quad dx=\frac{3}{2}\cos\theta\,d\theta \\
&=3\int \frac{\cos^2\theta}{\sin\theta}\,d\theta \\
&=3\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta \\
&=3\int \csc\theta-\sin\theta\,d\theta \\
&=-3\ln|\csc\theta+\cot\theta|+3\cos\theta+C \\
&=-3\ln\left|\frac{3}{2x}+\frac{\sqrt{9-4x^2}}{2x}\right|+\frac{3\sqrt{9-4x^2}}{3}+C
\end{aligned}$$

6. More trig substitution practice:

(a) $\displaystyle\int \frac{dx}{(x^2+9)^{3/2}}$

Solution: Use $x=3\tan\theta$

$$\int \frac{dx}{(x^2+9)^{3/2}} = \frac{x}{9\sqrt{x^2+9}}+C$$

(b) $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\,dx = 2\tan\theta-\theta+C = \sqrt{25x^2-4}-2\cos^{-1}\left(\frac{2}{5x}\right)+C$

Solution: Use $x=\frac{2}{5}\sec\theta$.

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx = 2\tan\theta-\theta+C = \sqrt{25x^2-4}-2\cos^{-1}\left(\frac{2}{5x}\right)+C$$

(c) $\displaystyle\int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,dx$

Solution: Use $x=(2/5)\sec\theta$.

$$\int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,dx = 2\tan\theta-\theta\Big|_{\theta=0}^{\theta=\pi/3} = 2\sqrt{3}-\frac{2\pi}{3}$$