Math 132: Discussion Session: Week 7

Directions: In groups of 3-4 students, work the problems on the following page. Below, list the members of your group and your answers to the specified questions. Turn **this paper** in at the end of class. You do not need to turn in the question page or your work.

Additional Instructions: It is okay if you do not completely finish all of the problems. Also, each group member should work through each problem, as similar problems may appear on the exam.

Scoring:

Correct answers	Grade
0–1	0%
2–3	80%
4–6	100%

Group Members:

7.2: Trigonometric Integrals.

(1) $\displaystyle\int \tan^6 x \sec^4 x \, dx =$

(2) $\displaystyle\int_0^{\pi/2} \cos^5(x) \, dx =$

(3) $\displaystyle\int_0^{\pi/6} \sin(3x)\cos(5x) \, dx =$

(4) $\displaystyle\int \cos^2(\sin t)\cos t \, dt =$

(5) $\displaystyle\int \cot^3 x \csc^3 x \, dx =$

(6) $\displaystyle\int \sec^3(x) \, dx =$

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7.2: Trigonometric Integrals. Compute the following integrals

(1) $\displaystyle\int \tan^6 x \sec^4 x \, dx.$

Solution: In this one, we might try either the substitution $u = \tan x$ or $u = \sec x$. With $u = \tan x$, we have $du = \sec^2 x \, dx$. We can write our integral as

$$\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x (\sec^2 x \, dx)$$

Is it possible to write $\tan^6 x \sec^2 x$ in terms of $u = \tan x$? Yes, we know that $\sec^2 x = \tan^2 x + 1 = u^2 + 1$. Thus,

$$\begin{aligned}
\int \tan^6 x \sec^4 x \, dx &= \int \tan^6 x \sec^2 x (\sec^2 x \, dx) \\
&= \int u^6 (u^2 + 1) \, du \\
&= \int (u^8 + u^6) \, du \\
&= \tfrac{1}{9} u^9 + \tfrac{1}{7} u^7 + C \\
&= \boxed{\tfrac{1}{9} \tan^9 x + \tfrac{1}{7} \tan^7 x + C.}
\end{aligned}$$

(2) $\displaystyle\int_0^{\pi/2} \cos^5(x) \, dx.$

Solution: We can split off a $\cos x \, dx = d(\sin x)$, suggesting substituting $u = \sin x$. The remaining part is $\cos^4 x$, which we can rewrite in terms of $\sin x$ by using

$$\cos^4 x = \left(\cos^2 x\right)^2 = \left(1 - \sin^2 x\right)^2.$$

We have

$$\begin{aligned}
\int \cos^5 x \, dx &= \int \cos^4 x \cdot \cos x \, dx \\
&= \int \left(1 - \sin^2 x\right)^2 \, d(\sin x) \\
&= \int \left(1 - u^2\right)^2 \, du \\
&= \int \left(1 - 2u^2 + u^4\right) \, du \\
&= u - \tfrac{2}{3} u^3 + \tfrac{1}{5} u^5 + C.
\end{aligned}$$

We're doing a definite integral, so we can stop here and substitute in the bounds. When $x = 0$, $u = \sin x = 0$. When $x = \pi/2$, $u = \sin x = 1$. Thus,

$$\int_0^{\pi/2} \cos^5 x \, dx = \left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = \tfrac{1}{3} + \tfrac{1}{5} = \boxed{\frac{8}{15}}.$$

(3) $\displaystyle\int_0^{\pi/6} \sin(3x) \cos(5x) \, dx.$

Solution: For this one, we should use the formula

$$\sin A \cos B = \tfrac{1}{2}(\sin(A + B) + \sin(A - B)).$$

In this case, this formula tells us that

$$\sin 3x \cos 5x = \tfrac{1}{2}(\sin 8x + \sin(-2x)) = \tfrac{1}{2}(\sin 8x - \sin 2x).$$

Thus,

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \tfrac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= \tfrac{1}{2}\left(-\tfrac{1}{8}\cos 8x + \tfrac{1}{2}\cos 2x\right) + C \\ &= \tfrac{1}{4}\cos 2x - \tfrac{1}{16}\cos 8x + C.\end{aligned}$$

We're not done yet, because we haven't answered the original question. We compute that

$$\begin{aligned}\int_0^{\pi/6} \sin 3x \cos 5x &= \left(\tfrac{1}{4}\cos\tfrac{\pi}{3} - \tfrac{1}{16}\cos\tfrac{4\pi}{3}\right) - \left(\tfrac{1}{4}\cos 0 - \tfrac{1}{16}\cos 0\right) \\ &= \tfrac{1}{4}\cdot\tfrac{1}{2} - \tfrac{1}{16}\cdot\left(-\tfrac{1}{2}\right) - \tfrac{1}{4} + \tfrac{1}{16} \\ &= \tfrac{1}{8} + \tfrac{1}{32} - \tfrac{1}{4} + \tfrac{1}{16} \\ &= \tfrac{1}{32}(4 + 1 - 8 + 2) = \boxed{-\frac{1}{32}}.\end{aligned}$$

(4) $\displaystyle\int \cos^2(\sin t)\cos t \, dt.$

Solution: This problem is unusual because there is a trigonometric function that is plugged into another trigonometric function, but that actually makes it easier to solve, because there is only one reasonable first step: substituting $u = \sin t$, $du = \cos t \, dt$. We find that

$$\int \cos^2(\sin t)\cos t \, dt = \int \cos^2 u \, du.$$

The integral $\int \sin^2 x \, dx$ appears in the book, and this one is pretty much the same. We use the formula $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$ and compute that

$$\int \cos^2 u \, du = \tfrac{1}{2}\int (1 + \cos 2u) \, du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C.$$

Of course, we're not done, because we haven't answered the question. To finish the problem, we substitute back in using $u = \sin t$.

$$\int \cos^2(\sin t)\cos t \, dt = \boxed{\tfrac{1}{2}\sin t + \tfrac{1}{4}\sin(2\sin t) + C.}$$

(5) $\displaystyle\int \cot^3 x \csc^3 x \, dx.$

Solution: There are two reasonable choices for substitution, either $u = \cot x$, $du = -\csc^2 x \, dx$ or $u = \csc x$, $du = -\cot x \csc x \, dx$. The first way, $u = \cot x$, leaves us with $\cot^3 x \csc x$, which is difficult to write in terms of $\cot x$. The second way, $u = \csc x$, leaves us with $\cot^2 x \csc^2 x$, which is more promising thanks to the identity $1 + \cot^2 x = \csc^2 x$. Using $u = \csc x$, $du = -\cot x \csc x$, we compute

$$\begin{aligned}\int \cot^3 x \csc^3 x \, dx &= -\int \cot^2 x \csc^2 x \cdot (-\cot x \csc x \, dx) \\ &= -\int (\csc^2 x - 1)\csc^2 x \cdot d(\csc x) \\ &= -\int (u^2 - 1)u^2 \, du \\ &= \int (u^2 - u^4) \, du \\ &= \tfrac{1}{3}u^3 - \tfrac{1}{5}u^5 + C \\ &= \boxed{\tfrac{1}{3}\csc^3 x - \tfrac{1}{5}\csc^5 x + C.}\end{aligned}$$

(6) $\displaystyle\int \sec^3(x) \, dx.$

Solution: This problem is one of the examples in the book and is tricky. If we do $u = \tan x$, then $du = \sec^2 x\,dx$, leaving us with one $\sec x$ left. But there's not a good way to write $\sec x$ in terms of u; we'd get $\sec x = \sqrt{u^2+1}$, which is not easy to work with. If we do $u = \sec x$, we get $du = \tan x \sec x\,dx$, which is also not that helpful.

We could try integration by parts, splitting it up $\sec^3 x = \sec x \cdot \sec^2 x$. The integral of $\sec x$ is complicated, but the integral of $\sec^2 x$ is just $\tan x$, so we set up the integration by parts as

$$\begin{aligned} f(x) &= \sec x, & g'(x) &= \sec^2 x, \\ f'(x) &= \tan x \sec x, & g(x) &= \tan x. \end{aligned}$$

Thus,

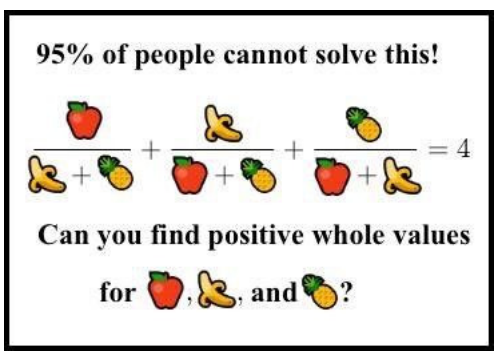
$$\begin{aligned} \int \sec^3 x\,dx &= \int fg'\,dx \\ &= fg - \int f'g\,dx \\ &= \sec x \tan x - \int \tan^2 x \sec x\,dx \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx \\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx. \end{aligned}$$

We're now in a nicer situation because we can solve the equation we found for $\int \sec^3 x\,dx$ to find that

$$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \int \sec x\,dx\right).$$

We're still not done, but the problem we need to solve now, $\int \sec x\,dx$, is easier than the original problem, $\int \sec^3 x\,dx$, so we've made progress.

However, $\int \sec x\,dx$ is still quite tricky. If you can solve it on your own and could go back in time to the mid-1600s, you'd be a cutting-edge mathematician. But you can still be a cutting-edge mathematician regardless; catching up on 350 years of work is easier than it seems, and you'll be solving deceptively hard fruit memes in no time.



In any case, back to the problem at hand, you can look up that $\int \sec x\,dx = \ln|\sec x + \tan x|$. You could also compute it with the substitution $u = \sec x + \tan x$, which gives

$$du = (\tan x \sec x + \sec^2 x)\,dx = (\sec x + \tan x)\sec x\,dx.$$

We multiply and divide by $\sec x + \tan x$ in order to get a $(\sec x + \tan x)\sec x\,dx$ factor, and we find that

$$\begin{aligned} \int \sec x\,dx &= \int \frac{1}{\sec x + \tan x}(\sec x + \tan x)\sec x\,dx \\ &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C. \end{aligned}$$

In any case, we get a final answer of

$$\boxed{\int \sec^3 x\,dx = \tfrac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C.}$$

Another avenue of attack for this problem is to write

$$\int \sec^3 x\,dx = \int \frac{1}{\cos^3 x}\,dx = \int \frac{1}{\cos^4 x}\cos x\,dx.$$

The reason to do this is that now we have a $\cos x\,dx$, which let us do the substitution $u = \sin x$, $du = \cos x\,dx$. Also, fortunately for us, we can rewrite $\cos^4 x$ in terms of sine using

$$\cos^4 x = \left(\cos^2 x\right)^2 = \left(1 - \sin^2 x\right)^2.$$

Thus,

$$\int \sec^3 x\,dx = \int \frac{1}{\cos^4 x}\cos x\,dx = \int \frac{1}{\left(1-u^2\right)^2}\,du.$$

Once again, we have converted an hard integral involving trigonometric functions into an integral that's quite a bit simpler. Unfortunately, the techniques needed to do this integral are in section 7.4, which we haven't covered yet. The ideas in that section give us the power to discover that

$$\frac{1}{\left(1-u^2\right)^2} = \frac{1}{4}\left(\frac{1}{1-u} + \frac{1}{(1-u)^2} + \frac{1}{1+u} + \frac{1}{(1+u)^2}\right).$$

We can do each of these integrals by substitution either $v = 1-u$ or $v = 1+u$. Careful with the signs! We integrate, simplify, and substitute to obtain

$$\begin{aligned}
\int \frac{1}{\left(1-u^2\right)^2}\,du &= \frac{1}{4}\left(-\ln|1-u| + \frac{1}{1-u} + \ln|1+u| - \frac{1}{1+u}\right) + C \\
&= \frac{1}{4}\left(\frac{2u}{1-u^2} + \ln\left|\frac{1+u}{1-u}\right|\right) + C \\
&= \frac{1}{4}\left(\frac{2\sin x}{1-\sin^2 x} + \ln\left|\frac{1+\sin x}{1-\sin x}\right|\right) \\
&= \frac{1}{4}\left(\frac{2\sin x}{\cos^2 x} + \ln\left|\frac{1+\sin x}{1-\sin x}\right|\right) \\
&= \boxed{\tfrac{1}{2}\sec x \tan x + \tfrac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right|.}
\end{aligned}$$

This looks almost the same as our previous answer, but not quite. Before, we had a $\frac{1}{2}\ln|\sec x + \tan x|$ term. Now, we have a $\frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$ term. But, in fact, these expressions are equal. We can see that by taking the first expression and rewriting it, making it look more and more like the second expression with each step.

$$\begin{aligned}
\tfrac{1}{2}\ln|\sec x + \tan x| &= \tfrac{1}{2}\ln\left|\frac{1+\sin x}{\cos x}\right| \\
&= \tfrac{1}{4}\cdot 2\ln\left|\frac{1+\sin x}{\cos x}\right| \\
&= \tfrac{1}{4}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right| \\
&= \tfrac{1}{4}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| \\
&= \tfrac{1}{4}\ln\left|\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}\right| \\
&= \tfrac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right|.
\end{aligned}$$