Integration by Parts:
\[ \int u \, dv = uv - \int v \, du \]

Warm-up Problems

1. **Clicker** The integral below can be done with integration by parts:
\[ \int (\ln x)^2 \, dx \]
What is a good choice for \( u \)?
(a) 1  
(b) \( \ln x \)  
(c) \( (\ln x)^2 \) **Correct**  
(d) \( dx \)  
(e) It can’t be done by integration by parts.

**Solution:** If \( u = (\ln x)^2 \) then \( du = 2(\ln x) \cdot \frac{1}{x} \cdot dx \) and \( v = x \) and the integral becomes
\[ \int (\ln x)^2 \, dx = x(\ln x)^2 - \int 2 \ln x \cdot \frac{1}{x} \cdot x \, dx \]
\[ = x(\ln x)^2 - \int 2 \ln x \, dx \]

Integration by parts is necessary a second time. You should get:
\[ x(\ln x)^2 - 2x \ln x + 2x + C \]

2. In Problem [ ] what is \( v \)?

**Solution:** \( x \).

Class Problems

3. \( \int e^x \cos x \, dx \)

**Solution:** Trickier, use integration by parts twice and notice the pattern.
\[ \frac{1}{2} e^x (\cos x + \sin x) + C \]

The trick for these is the following:
\[ I = \int e^x \cos x \, dx \quad (u = e^x, \, du = \cos x \, dx) \]
\[ = e^x \sin x - \int e^x \sin x \, dx \quad (u = e^x, \, dv = \sin x \, dx) \]
\[ = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \]
\[ I = e^x \sin x + e^x \cos x - I \]
\[ 2I = e^x \sin x + e^x \cos x \]
\[ I = \frac{1}{2} (e^x \sin x + e^x \cos x) \]

Be sure to try this again but with \( u = \cos x \) and \( dv = e^x \, dx \). It should work just the same.
4. \( \int e^x \sin x \, dx \)
   **Solution:** Integrate by parts twice:
   \[
   I = -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + \left( e^x \sin x - \int e^x \sin x \, dx \right)
   \]
   \[
   I = \frac{1}{2}(-e^x \cos x + e^x \sin x)
   \]

5. \( \int e^{3x} \cos 4x \, dx \)
   **Solution:** \( \frac{1}{25}e^{3x}(4 \sin 4x + 3 \cos 4x) + C \)

6. \( \int x^2 e^{3x} \, dx \)
   **Solution:** \( \frac{1}{27}e^{3x}(9x^2 - 6x + 2) + C \)

7. \( \int x^3 e^{3x} \, dx \)
   **Solution:** \( \frac{1}{27}e^{3x}(9x^3 - 9x^2 + 6x - 2) + C \)

8. \( \int x \sin x \, dx \)
   **Solution:** One option is to use \( u = x \) and \( dv = \sin x \, dx \). Another method is to use a trig identity and transform \( \sin x \cos x = \frac{1}{2} \sin 2x \) and then proceed as normal. In any case your solution might look like one of the following:
   \[
   \frac{1}{4}x \sin^2 x - \frac{1}{4}x \cos^2 x + \frac{1}{8} \sin 2x + C \quad \text{OR} \quad \frac{1}{8} \sin 2x - \frac{1}{4}x \cos 2x + C
   \]

9. \( \int x \ln x \, dx \)
   **Solution:** Use \( u = \ln x \) and \( dv = x \, dx \).
   \[ \int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C \]

10. \( \int \frac{\ln x}{x^2} \, dx \)
    **Solution:** You can do a substitution of \( w = \ln x \) and transform into a \( w \) integral (TRY IT!). Or you can use \( u = \ln x \) and \( dv = \frac{1}{x^2} \, dx \).
    \[ \int \frac{\ln x}{x^2} \, dx = -\ln x - \frac{1}{10x^4} + C \]

11. \( \int \arctan x \, dx \)
    **Solution:** use parts with \( u = \arctan x \) and \( dv = dx \).
    \[
    \int \arctan x \, dx = x \arctan x - \int \frac{x}{x^2 + 1} \, dx
    \]
    \[
    = x \arctan x - \frac{1}{2} \ln(x^2 + 1) + C
    \]

12. \( \int 2x \arctan x \, dx \)
    **Solution:** There is an algebraic trick at the end (a method which we will formalize later as **Partial Fractions**). Use parts with \( u = \arctan x \) and \( dv = 2x \, dx \).
    \[
    \int 2x \arctan x \, dx = x^2 \arctan x - \int \frac{x^2}{x^2 + 1} \, dx
    \]
    \[
    = x^2 \arctan x - \int \frac{x^2 + 1 - 1}{x^2 + 1} \, dx
    \]
    \[
    = x^2 \arctan x - \int \frac{x^2 + 1}{x^2 + 1} \, dx + \int \frac{1}{x^2 + 1} \, dx
    \]
    \[
    = x^2 \arctan x - x + \arctan x + C
    \]

13. **Clicker** Choose the correct \( u \) if using integration by parts: \( \int (\text{Polynomial}(x)) e^x \, dx \)
    (a) \( u = 1 \) (b) \( u = \text{Polynomial}(x) \) **Correct** (c) \( u = e^x \) (d) \( u = \sin x \) (e) \( u = \cos x \)

14. **Clicker** Choose the correct \( u \) if using integration by parts: \( \int (\text{Polynomial}(x)) \sin x \, dx \)
    (a) \( u = 1 \) (b) \( u = \text{Polynomial}(x) \) **Correct** (c) \( u = e^x \) (d) \( u = \sin x \) (e) \( u = \cos x \)
15. **Clicker** Choose the correct $u$ if using integration by parts: $\int (\text{Polynomial}(x)) \cos x \, dx$
   (a) $u = 1$  (b) $u = \text{Polynomial}(x)$  **Correct**  (c) $u = e^x$  (d) $u = \sin x$  (e) $u = \cos x$

16. **Clicker** Choose the correct $u$ if using integration by parts: $\int e^x \sin x \, dx$
   (a) $u = 1$  (b) $u = e^x$  **Correct**  (c) $u = \sin x$  **Correct**  (d) $u = \cos x$  (e) $u = \ln x$

17. **Clicker** Choose the correct $u$ if using integration by parts: $\int e^x \cos x \, dx$
   (a) $u = 1$  (b) $u = e^x$  **Correct**  (c) $u = \sin x$  (d) $u = \cos x$  **Correct**  (e) $u = \ln x$

18. **Clicker** Choose the correct $u$ if using integration by parts: $\int \ln x \, dx$
   (a) $u = 1$  (b) $u = e^x$  (c) $u = \sin x$  (d) $u = \cos x$  (e) $u = \ln x$  **Correct**

**Lecture Notes:** Integration by parts, the “normal” types you will see and you have to have “down cold”:

- $\int (\text{Polynomial}(x)) e^x \, dx$
- $\int (\text{Polynomial}(x)) \sin x \, dx$
- $\int (\text{Polynomial}(x)) \cos x \, dx$
- $\int e^x \sin x \, dx$
- $\int e^x \cos x \, dx$
- $\int \ln x \, dx$

19. $\int \sin x \cos x \, dx$

   **Solution:** This is a simple substitution. You can use either $u = \sin x$ or $u = \cos x$. In either case you get

   $$\int \sin x \cos x \, dx = \int u \, du \quad (u = \sin x)$$
   $$= \frac{u^2}{2} + C = \frac{1}{2} \sin^2 x + C$$

   $$\int \sin x \cos x \, dx = - \int u \, du \quad (u = \cos x)$$
   $$= -\frac{u^2}{2} + C = -\frac{1}{2} \cos^2 x + C$$

   How can this be????