Math 132 - September 29, 2017

Solutions

Sections Covered: 6.5, 7.1

Average Value and MVT for Integrals:

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Integration by Parts:

$$\int u\,dv = uv - \int v\,du$$

Warm-up Problems

1. $\boxed{\textbf{Clicker}}$ Suppose $f(x)$ is continuous on $[-1,1]$ and $\int_{-1}^{1} f(x)\,dx = \pi$. Which of the following values does f have to take on? (which of the following is a possible y value?)

   (a) 1
   (b) -1
   (c) π
   (d) $\pi/2$
   (e) Not enough information

   Solution: By the MVT for integrals, we must have an x value $c \in [-1,1]$ such that $f(c) = f_{\text{ave}} = \frac{\pi}{2}$. Thus, f must have this y value.

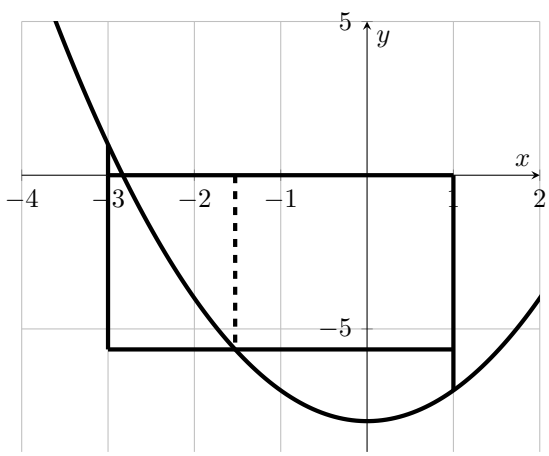
2. Let $f(x) = x^2 - 8$.

   (a) Find the average value of f on $[-3,1]$.
   Solution:

   $$f_{\text{ave}} = \frac{1}{4}\int_{-3}^{1}(x^2-8)\,dx = -\frac{17}{3}$$

   (b) Find c in $[-1,3]$ such that $f(c)$ equals the average value. (This is the value guaranteed by the MVT for integrals.)
   Solution: Solve $f(x) = -\frac{17}{3}$ to get $x = \pm\sqrt{\frac{7}{3}}$. The $+$ value is outside our interval so we get $c = -\sqrt{\frac{7}{3}}$.

   (c) Draw a graph of $f(x)$ with the average value labeled, the point c labeled and draw an appropriate rectangle.
   Solution:



3. Find the average value of $f(x) = |x^2 - 2|$ on $[0,2]$.

Solution:

$$\begin{aligned} f_{\text{ave}} =& \frac{1}{2}\int_0^2 f(x)\ dx \\ =& \frac{1}{2}\int_0^{\sqrt{2}} -(x^2-2)\ dx + \frac{1}{2}\int_{\sqrt{2}}^2 x^2 - 2\ dx \\ =& \frac{1}{2}\left(\frac{4\sqrt{2}}{3} + \frac{4}{3}(\sqrt{2}-1)\right) = \frac{2}{3}\left(2\sqrt{2}-1\right) \end{aligned}$$

Class Problems

4. $\boxed{\textbf{Clicker}}$ If u and v and functions and "d" means derivative, the product rule can be written as

$$d(uv) = v\ du + u\ dv$$

Write this in integral form.

(a) $uv = vu + uv$
(b) $\int uv = \int v\ du + \int u\ dv$
(c) $uv = \int v\ du + \int u\ dv$ $\boxed{\textbf{Correct}}$
(d) $\int d(uv) = \int (v\ du - u\ dv)$
(e) $uv = \int v\ du - \int u\ dv$

Lecture Notes: Lets start with the product rule and turn it into an integration rule:

$$\begin{aligned} & d(uv) = v\ du + u\ dv \\ & \int d(uv) = \int (v\ du + u\ dv) \\ & uv = \int v\ du + \int u\ dv \\ & \int u\ dv = uv - \int v\ du \qquad \textbf{Integration by Parts} \end{aligned}$$

Just like with substitution, keeping track of things in an orderly way is key. If you aren't organized you will make mistakes. I recommend you write $u =$ and $dv =$ to start with and follow it exactly like I do it at the board.

Examples:

- $\int xe^x\ dx$: $u = x, dv = e^x\ dx$
- $\int x^2e^x\ dx$: $u = x^2, dv = e^x\ dx$

5. Use integration by parts to compute the integral. Use the given u and dv.

(a) $\displaystyle\int \ln x\ dx$
$u = \ln x$, $dv = dx$
Solution: Then $du = \frac{1}{x}\ dx$ and $v = x$
$\int \ln x\ dx = x\ln x - \int x \cdot \frac{1}{x}\ dx = x\ln x - x + C$

(b) $\displaystyle\int x\cos x\ dx$
$u = x$, $dv = \cos x\ dx$
Solution: Then $du = dx$ and $v = \sin x$
$\int x\cos x\ dx = x\sin x - \int \sin x\ dx = x\sin x + \cos x + C$

(c) $\displaystyle\int x\cos x\ dx$
$u = \cos x$, $dv = x\ dx$
Solution: Then $du = -\sin x\ dx$ and $v = \frac{1}{2}x^2$
$\int x\cos x\ dx = \frac{1}{2}x^2\cos x - \int \frac{1}{2}x^2(-\sin x)\ dx =???$

6. Use integration by parts to compute: $\int_1^4 xe^x \, dx$

   Solution: One key with definite integrals, you get to choose when to substitute limits. I usually just wait until the end, but you can substitute in the middle.

$$\begin{aligned}\int_1^4 xe^x \, dx =& xe^x\Big|_1^4 - \int_1^4 e^x \, dx \qquad (u = x, dv = e^x \, dx)\\ =&(4e^4 - e^1) - \int_1^4 e^x \, dx = 3e^4\end{aligned}$$

7. Use integration by parts to find the indefinite integral

   (a) $\int x^2 e^{3x} \, dx$
   Solution: Use integration by parts twice. Start with $u = x^2$.
   $\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$

   (b) $\int x^2 \sin(10x) \, dx$
   Solution: Integration by parts twice, use $u = x^2$.
   $-\frac{x^2}{10}\cos 10x + \frac{x}{50}\sin 10x + \frac{1}{500}\cos 10x + C$

   (c) $\int e^x \cos x \, dx$
   Solution: Trickier, use integration by parts twice and notice the pattern.
   $\frac{1}{2}e^x(\cos x + \sin x) + C$

   (d) $\int (\ln x)^2 \, dx$
   Solution: Integration by parts with $u = (\ln x)^2$
   $x(\ln x)^2 - 2x\ln x + 2x + C$

   (e) $\int x^3 \cos(x^2) \, dx$
   Solution: Integration by parts 3 times. Ugh.
   $\frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C$

8. Use integration by parts to find the indefinite integral: Use $u = e^x$ and $dv = \sin x \, dx$.
   $\int e^x \sin x \, dx$

   Solution: Integrate by parts twice: $= -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + \left(e^x \sin x - \int e^x \sin x \, dx\right)$

9. More integration by parts practice

   (a) $\int e^{3x} \cos 4x \, dx$
   Solution: $\frac{1}{25}e^{3x}(4\sin 4x + 3\cos 4x) + C$

   (b) $\int x^2 e^{3x} \, dx$
   Solution: $\frac{1}{27}e^{3x}(9x^2 - 6x + 2) + C$

   (c) $\int x^3 e^{3x} \, dx$
   Solution: $\frac{1}{27}e^{3x}(9x^3 - 9x^2 + 6x - 2) + C$

   (d) $\int \arctan x \, dx$
   Solution: Use $u = \arctan x$
   $x\arctan x - \frac{1}{2}\ln(1 + x^2) + C$

   (e) $\int x \sin x \cos x \, dx$
   Solution: This is a bit messy. Probably the easiest option is to use $u = x$ and then worry about finding v. If you do it this way, you can use $\sin x \cos x = \frac{1}{2}\sin 2x$. Then it becomes easy. Here is one form of the solution:
   $\frac{1}{4}x\sin^2 x - \frac{1}{4}x\cos^2 x + \frac{1}{8}\sin 2x + C$
   This is equivalent to $\frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x + C$