Math 132 - September 29, 2017
Solutions

Sections Covered: 6.5, 7.1

Average Value and MVT for Integrals:

\[ f(c) = f_{\text{ave}} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \]

Integration by Parts:

\[ \int u \, dv = uv - \int v \, du \]

Warm-up Problems

1. **Clicker** Suppose \( f(x) \) is continuous on \([-1, 1]\) and \( \int_{-1}^{1} f(x) \, dx = \pi \). Which of the following values does \( f \) have to take on? (which of the following is a possible \( y \) value?)

(a) 1
(b) −1
(c) \( \pi \)
(d) \( \pi / 2 \)
(e) Not enough information

**Solution:** By the MVT for integrals, we must have an \( x \) value \( c \in [-1, 1] \) such that \( f(c) = f_{\text{ave}} = \frac{\pi}{2} \). Thus, \( f \) must have this \( y \) value.

2. Let \( f(x) = x^2 - 8 \).

(a) Find the average value of \( f \) on \([-3, 1]\).

**Solution:**

\[ f_{\text{ave}} = \frac{1}{4} \int_{-3}^{1} (x^2 - 8) \, dx = -\frac{17}{3} \]

(b) Find \( c \) in \([-1, 3]\) such that \( f(c) \) equals the average value. (This is the value guaranteed by the MVT for integrals.)

**Solution:** Solve \( f(x) = -\frac{17}{3} \) to get \( x = \pm \sqrt{\frac{7}{3}} \). The + value is outside our interval so we get \( c = -\sqrt{\frac{7}{3}} \).

(c) Draw a graph of \( f(x) \) with the average value labeled, the point \( c \) labeled and draw an appropriate rectangle.

**Solution:**

![Graph](image)

3. Find the average value of \( f(x) = |x^2 - 2| \) on \([0, 2]\).
Solution:
\[ f_{\text{ave}} \frac{1}{2} \int_{0}^{2} f(x) \, dx \]
\[ = \frac{1}{2} \int_{0}^{\sqrt{2}} -x^2 - 2 \, dx + \frac{1}{2} \int_{\sqrt{2}}^{2} x^2 - 2 \, dx \]
\[ = \frac{1}{2} \left( \frac{4\sqrt{2}}{3} + \frac{4}{3}(\sqrt{2} - 1) \right) = \frac{2}{3}(2\sqrt{2} - 1) \]

Class Problems
4. **Clicker** If \( u \) and \( v \) are functions and “\( d \)” means derivative, the product rule can be written as
\[ d(uv) = v \, du + u \, dv \]
Write this in integral form.
(a) \( uv = vu + uv \)
(b) \( \int uv = \int v \, du + \int u \, dv \)
(c) \( uv = \int v \, du + \int u \, dv \) **Correct**
(d) \( \int d(uv) = \int (v \, du - u \, dv) \)
(e) \( uv = \int v \, du - \int u \, dv \)

**Lecture Notes:** Let's start with the product rule and turn it into an integration rule:
\[ d(uv) = v \, du + u \, dv \]
\[ \int d(uv) = \int (v \, du + u \, dv) \]
\[ uv = \int v \, du + \int u \, dv \]
\[ \int u \, dv = uv - \int v \, du \] **Integration by Parts**

Just like with substitution, keeping track of things in an orderly way is key. If you aren’t organized you will make mistakes. I recommend you write \( u = \) and \( dv = \) to start with and follow it exactly like I do it at the board.

**Examples:**
- \( \int xe^x \, dx \): \( u = x, \, dv = e^x \, dx \)
- \( \int x^2 e^x \, dx \): \( u = x^2, \, dv = e^x \, dx \)

5. Use integration by parts to compute the integral. Use the given \( u \) and \( dv \).
(a) \( \int \ln x \, dx \)
\( u = \ln x, \, dv = dx \)
**Solution:** Then \( du = \frac{1}{x} \, dx \) and \( v = x \)
\[ \int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x + C \]
(b) \( \int x \cos x \, dx \)
\( u = x, \, dv = \cos x \, dx \)
**Solution:** Then \( du = dx \) and \( v = \sin x \)
\[ \int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C \]
(c) \( \int x \cos x \, dx \)
\( u = \cos x, \, dv = x \, dx \)
**Solution:** Then \( du = -\sin x \, dx \) and \( v = \frac{1}{2}x^2 \)
\[ \int x \cos x \, dx = \frac{1}{2}x^2 \cos x - \int \frac{1}{2}x^2 (-\sin x) \, dx = ??? \]
6. Use integration by parts to compute: \( \int_1^4 xe^x \, dx \)

**Solution:** One key with definite integrals, you get to choose when to substitute limits. I usually just wait until the end, but you can substitute in the middle.

\[
\int_1^4 xe^x \, dx = \left[ xe^x \right]_1^4 - \int_1^4 e^x \, dx \quad (u = x, dv = e^x \, dx)
\]

\[
= (4e^4 - e^1) - \int_1^4 e^x \, dx = 3e^4
\]

7. Use integration by parts to find the indefinite integral

(a) \( \int x^2 e^{3x} \, dx \)

**Solution:** Use integration by parts twice. Start with \( u = x^2 \).

\[
\frac{1}{3} x^2 e^{3x} - \frac{2}{9} xe^{3x} + \frac{2}{27} e^{3x} + C
\]

(b) \( \int x^2 \sin(10x) \, dx \)

**Solution:** Integration by parts twice, use \( u = x^2 \).

\[
-\frac{1}{10} x^2 \cos 10x + \frac{1}{100} x \sin 10x + \frac{1}{1000} \cos 10x + C
\]

(c) \( \int e^x \cos x \, dx \)

**Solution:** Trickier, use integration by parts twice and notice the pattern.

\[
\frac{1}{2} e^x (\cos x + \sin x) + C
\]

(d) \( \int \ln x^2 \, dx \)

**Solution:** Integration by parts with \( u = \ln x^2 \)

\[
x \ln x^2 - 2x \ln x + 2x + C
\]

(e) \( \int e^x \sin x \, dx \)

**Solution:** Integration by parts 3 times. Ugh.

\[
\frac{1}{2} e^x (\cos x + \sin x) + C
\]

8. Use integration by parts to find the indefinite integral: Use \( u = e^x \) and \( dv = \sin x \, dx \).

\[ \int e^x \sin x \, dx \]

**Solution:** Integrate by parts twice: \( -e^x \cos x + \int e^x \cos x \, dx = -e^x \cos x + (e^x \sin x - \int e^x \sin x \, dx) \)

9. More integration by parts practice

(a) \( \int e^{3x} \cos 4x \, dx \)

**Solution:** \( \frac{1}{25} e^{3x} (4 \sin 4x + 3 \cos 4x) + C \)

(b) \( \int x^2 e^{3x} \, dx \)

**Solution:** \( \frac{1}{27} e^{3x} (9x^2 - 6x + 2) + C \)

(c) \( \int x^3 e^{3x} \, dx \)

**Solution:** \( \frac{1}{27} e^{3x} (9x^3 - 9x^2 + 6x - 2) + C \)

(d) \( \int \arctan x \, dx \)

**Solution:** Use \( u = \arctan x \)

\[
x \arctan x - \frac{1}{2} \ln(1 + x^2) + C
\]

(e) \( \int x \sin x \cos x \, dx \)

**Solution:** This is a bit messy. Probably the easiest option is to use \( u = x \) and then worry about finding \( v \). If you do it this way, you can use \( \sin x \cos x = \frac{1}{2} \sin 2x \). Then it becomes easy. Here is one form of the solution:

\[
\frac{1}{2} x \sin^2 x - \frac{1}{4} x \cos^2 x + \frac{1}{2} \sin 2x + C
\]

This is equivalent to \( \frac{1}{8} \sin 2x - \frac{1}{4} \cos 2x + C \)