Practice Exam 3 Solutions

1) $a_n = n^3 \ln(1 + 3/n^3)$

$$\lim_{n\to\infty} n^3 \ln(1+3/n^3) = \lim_{x\to\infty} x^3 \ln(1+3/x^3) \quad \infty\cdot 0$$

$$= \lim_{x\to\infty} \frac{\ln(1+3/x^3)}{1/x^3} \quad 0/0$$

$$= \lim_{x\to\infty} \frac{3\cdot(-3x^{-4})\cdot \frac{1}{1+3/x^3}}{-3x^{-4}}$$

$$= \lim_{x\to\infty} \frac{3}{1+3/x^3} = \boxed{3}$$

F

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2) $\frac{2^3}{7} + \frac{2^4}{7^2} + \frac{2^5}{7^3} + \frac{2^6}{7^4} + \dots$

$$= 2^2\left[2/7 + 2^2/7^2 + 2^3/7^3 + 2^4/7^4 + \dots\right]$$

$$= \sum_{n=1}^{\infty} 4\cdot[2/7]^n = \frac{4\cdot 2/7}{1-2/7} = \frac{8/7}{5/7} = \boxed{8/5}$$

F

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3) $\lim_{n\to\infty} \left(1-\frac{5}{n}\right)^{3n} = L$

$$\ln(L) = \lim_{x\to\infty} \ln\left[\left(1-\frac{5}{x}\right)^{3x}\right] = \lim_{x\to\infty} 3x \ln\left[1-\frac{5}{x}\right] \quad \infty\cdot 0$$

$$= \lim_{x\to\infty} \frac{3\ln[1-5/x]}{1/x} \quad 0/0$$

$$= \lim_{x\to\infty} \frac{3\cdot(5x^{-2})\frac{1}{1-5/x}}{-x^{-2}}$$

$$= -15$$

F

$\Rightarrow \ln(L) = -15$, so the limit $L = \boxed{e^{-15}}$

4) $\sum_{n=3}^{\infty} \frac{2}{n(n-1)}$

Telescoping $\frac{2}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1} \Rightarrow 2 = A(n-1) + Bn$

$= \frac{-2}{N} + \frac{2}{n-1}$ $\quad n=0 \quad A=-2$

$n=1 \quad B=2$

$S_3 = -\frac{2}{3} + \frac{2}{3-1} = 1 - 2/3$

$S_4 = -2/3 + 2/3{-1} - 2/4 + 2/3 = 1 - 2/4$

$\vdots\ S_N = 1 - 2/N$

$\boxed{\lim_{N\to\infty} S_N = 1}$ (D)

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(5) $a_n = \frac{-n^2 + \cos(3/n^2)}{3n^2 + \sqrt{4n^4 + 9n^3 + 1}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-n^2 + \cos(3/n^2)}{3n^2 + \sqrt{4n^4 + 9n^3 + 1}} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$

$= \lim_{n\to\infty} \frac{-1 + \cos(3/n^2)/n^2}{3 + \sqrt{4 + 9/n + 1/n^4}}$

$= \frac{-1}{3+\sqrt{4}} = \boxed{-\frac{1}{5}}$ (E)

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(6) $\sum_{n=0}^{\infty} \frac{5(-2)^{3n}}{3^{2n}} = \sum_{n=0}^{\infty} \frac{5\cdot(-8)^n}{9^n} = \sum_{n=0}^{\infty} 5 \cdot \left[\frac{-8}{9}\right]^n$

$= \frac{5}{1 + 8/9}$

$= \boxed{\frac{45}{17}}$ (B)

7 I. True by the n^{th} term Test for divergence

II. True, by the definition of $\sum a_n$ converging

III False. We need $a_n, b_n \geq 0$. For Example $a_n = -n$ & $b_n = \frac{1}{n^2}$ satisfy the given conditions but $\sum -n$ diverges.

IV False. We need $a_n, b_n \geq 0$. For Example $a_n = -n$ & $b_n = \frac{1}{n^2}$ satisfy the given conditions but $\sum \frac{1}{n^2}$ converges

D

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8 $S_N = 2 + N^2 - N = a_1 + a_2 + \dots + a_N$

$$\sum_{n=3}^{6} a_n = a_3 + a_4 + a_5 + a_6$$
$$= (a_1 + a_2 + \dots + a_6) - (a_1 + a_2)$$
$$= S_6 - S_2$$
$$= [2 + 36 - 6] - [2 + 4 - 2]$$
$$= 32 - 4$$
$$= \boxed{28}$$

C

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9 $a_n = \frac{n!}{n^n}$ $b_n = \frac{(2n-1)!}{(2n+1)!}$ $c_n = \frac{\ln(n^{100})}{n^2}$ $d_n = \frac{3^n}{2^n n^3}$

$a_n = \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n} \leq \frac{1}{n} \to 0$ and $a_n \geq 0$.

This implies $\lim_{n \to \infty} a_n = 0$ by the squeeze Theorem

$b_n = \frac{(2n-1)!}{(2n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdots (2n-1) 2n (2n+1)} = \frac{1}{2n(2n+1)}$

This implies $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{2n(2n+1)} = 0$

c_n $\lim_{n \to \infty} c_n = \lim_{x \to \infty} \frac{100 \ln x}{x^2} \overset{\infty/\infty}{=} \lim_{x \to \infty} \frac{100/x}{2x} = \lim_{x \to \infty} \frac{50}{x^2} = 0.$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{[3/2]^n}{n^3} = \infty$ b/c Exponentials grow Faster than powers of n.

Note: You can also apply L'Hopital's Rule 3 times.

$\therefore \{a_n\}, \{b_n\}, \& \{c_n\}$ converge. (E)

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10) I $\sum_{n=1}^{\infty} \underbrace{\frac{\ln n}{n^2-3n}}_{=a_n}$ compare to $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^{3/2}}}_{=b_n}$ p-series, $p>1$ so converges

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^2-3n}}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^{3/2}\ln n}{n^2-3n} = \lim_{n\to\infty} \frac{n^{1/2}\ln n}{n-3}$$

$$= \lim_{n\to\infty} \frac{n^{1/2}}{\sqrt{n-3}} \cdot \frac{\ln n}{\sqrt{n-3}} = 1 \cdot \lim_{x\to\infty} \frac{\ln x}{\sqrt{x-3}} = \lim_{x\to\infty} \frac{1/x}{1/2(x-3)^{-1/2}}$$

$$= \lim_{x\to\infty} \frac{2\sqrt{x-3}}{x} = 0$$

By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{\ln n}{n^2-3n}$ also converges

II $\sum_{n=1}^{\infty} \frac{n}{4^{-n}+5^{-n}}$

$\lim_{n\to\infty} \frac{n}{4^{-n}+5^{-n}} = \infty$ (b/c $4^{-n}, 5^{-n} \to 0$ and $n \to \infty$)

$\therefore$ The series diverges by the n^{th} term Test

III $\sum_{n=1}^{\infty} \frac{n-\cos n}{n^3}$ compare to $\sum_{n=1}^{\infty} 1/n^2$ converges b/c p-series with $p=2$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n-\cos n}{n^3}}{1/n^2} = \lim_{n\to\infty} \frac{n^3-n^2\cos n}{n^3} \frac{\cdot 1/n^3}{1/n^3}$$

$$= \lim_{n\to\infty} \frac{1-1/n\cos(n)}{1} = 1$$

$\therefore$ By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{n-\cos n}{n^3}$ converges

$\boxed{\text{So I \& III converge}}$ (F)

11

I. $\sum_{n=1}^{\infty} \frac{(\ln n)^{12}}{n^{9/8}}$ Limit compare to $\sum_{n=1}^{\infty} \frac{1}{n^{9/8-1/16}}$ converges b/c p-series w/ $p>1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{(\ln n)^{12}}{n^{9/8}}}{1/n^{9/8-1/16}} = \lim_{n\to\infty} \frac{(\ln n)^{12}}{n^{1/16}} = 0$$

here, you can do L'Hopital's Rule or just know that $\frac{\ln(n^a)}{n^b} \to 0 \quad a,b>0$

$\therefore$ By the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{(\ln n)^{12}}{n^{9/8}}$ also converges.

II $\sum_{n=1}^{\infty} 4^{1/n}$

$\lim_{n\to\infty} 4^{1/n} = 4^0 = 1 \neq 0$. So, the series diverges by the n^{th} term test

III $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^{3/2}-2n}$ Limit compare to $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges b/c $p=1/2 \leq 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n^2+1}}{n^{3/2}-2n}}{1/n^{1/2}} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n-2n^{1/2}} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{\sqrt{1+1/n^2}}{1-2/\sqrt{n}} = 1.$$

By the Limit Comparison Test $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^{3/2}-2n}$ also diverges

I only

B

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12 I.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+3}$ $\frac{1}{n+3}$ is positive, decreasing, & $\therefore \lim_{n\to\infty} \frac{1}{n+3} = 0$

So, the series converges by the alterating series test

II $\sum_{n=1}^{\infty} (-1)^n \left[\sqrt{4n^2+1} - 1\right]$

$$\lim_{n\to\infty} (-1)^n \sqrt{4n^2+1} - 1 \cdot \frac{\sqrt{4n^2+1}+1}{\sqrt{4n^2+1}+1} = \lim_{n\to\infty} \frac{(-1)^n 2n}{\sqrt{4n^2+1}+1}$$

some $\to 1$
some $\to -1$

$\therefore$ series diverges by the n^{th} term test.

$\sum_{n=1}^{\infty} (-1)^n [\arctan(1/n)]$ $\lim_{n\to\infty} \arctan(1/n) = 0$ & $\arctan(1/n) \geq 0$

B/c $\arctan x$ is increasing on $[0,\infty)$, $\arctan(1/x)$ is decreasing on $[0,\infty)$

$\therefore$ series converges by the Alternating Series Test   I & III   F

#13 $\sum_{n=1}^{\infty} \frac{n^2}{(n^3+2)^p}$ Limit compare to $\sum_{n=1}^{\infty} \frac{n^2}{n^{3p}} = \sum_{n=1}^{\infty} \frac{1}{n^{3p-2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2}{(n^3+2)^p}}{1/n^{3p-2}} = \lim_{n\to\infty} \frac{n^{3p}}{(n^3+2)^p} \cdot \frac{1/n^{3p}}{1/n^{3p}} = \lim_{n\to\infty} \frac{1}{(1+2/n^3)^p} = 1$$

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{(n^3+2)^p}$ & $\sum_{n=1}^{\infty} \frac{1}{n^{3p-2}}$ have the same behavior, i.e. converge if $3p-2 > 1 \Rightarrow 3p > 3 \Rightarrow p > 1$.   C

#14 $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}$

This is a geometric series with $r = \frac{x-2}{3}$. To converge, we need $-1 < \frac{x-2}{3} < 1 \Rightarrow -3 < x-2 < 3 \Rightarrow -1 < x < 5$, or $\boxed{(-1,5)}$ F

#15 I $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ no b/c $a_n = \frac{(-1)^n}{n}$ isn't ≥ 0

II $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ no b/c $\frac{2^x}{x!}$ isn't a well-defined function

III $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ yes $\frac{1}{x^2 \ln x}$ is positive, continuous, & decreasing

D

16

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}}$ Note $3^{\ln n} = e^{\ln 3 \cdot \ln n} = (e^{\ln n})^{\ln 3} = n^{\ln 3}$

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\ln 3}}$ which converges because it's a p-series with $p > 1$

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b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ Limit compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges b/c p-series, w/ $p = 3/2 > 1$.

$$\lim_{n\to\infty} \frac{\frac{\sin(1/n)}{\sqrt{n}}}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^{3/2}\sin(1/n)}{\sqrt{n}} = \lim_{n\to\infty} n\sin(1/n) \quad \infty \cdot 0$$

$$= \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \quad 0/0$$

$$= \lim_{x\to\infty} \frac{-1/x^2 \cos(1/x)}{-1/x^2} = 1.$$

By Limit Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ also converges.

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c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{\sqrt{n}}}$ Limit Compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{1}{e^{\sqrt{n}}}}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{e^{\sqrt{n}}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{2x}{1/2x^{-1/2}e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{4x^{3/2}}{e^{\sqrt{x}}} \quad \infty/\infty$$

$$= \lim_{x\to\infty} \frac{6x^{1/2}}{\cdot 1/2x^{-1/2}e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{12x}{e^{\sqrt{x}}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{12}{1/2x^{-1/2}e^{\sqrt{x}}}$$

use L'Hopital's Rule a bunch

$$= \lim_{x\to\infty} \frac{24x^{1/2}}{e^{\sqrt{x}}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{12x^{-1/2}}{1/2x^{-1/2}e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{24}{e^{\sqrt{x}}} = 0.$$

By the Limit Comparison Test, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges b/c it's a p-series w/ $p=2$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{\sqrt{n}}}$ also converges.

17

a) $\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n^3/3}$

Consider $\sum_{n=1}^{\infty} n^2 e^{-n^3/3}$ $f(x) = x^2 e^{-x^3/3}$ is positive & continuous on $[1, \infty)$

and $f'(x) = [2x - x^4] e^{-x^3/3} < 0$ on $[2, \infty)$

so f is decreasing.

$\therefore$ We can apply the integral Test.:

$$\int_2^{\infty} x^2 e^{-x^3/3} dx = \lim_{t \to \infty} \int_2^t x^2 e^{-x^3/3} dx = \lim_{t \to \infty} \int_{-8/3}^{-t^3/3} -e^u \, du = \lim_{t \to \infty} -e^u \Big|_{-8/3}^{-t^3/3}$$

$u = -x^3/3 \quad du = -x^2 dx$

$$= \lim_{t \to \infty} e^{-8/3} - e^{-t^3/3} = e^{-8/3}$$

Since the integral converges, $\sum_{n=2}^{\infty} n^2 e^{-n^3/3}$ converges as well.

Thus, $\sum_{n=2}^{\infty} (-1)^n n^2 e^{-n^3/3}$ $\boxed{\text{converges absolutely}}$

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b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$

Note $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ can be limit-compared to $\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\lim_{n \to \infty} \frac{\frac{1}{\sqrt{n^2+1}}}{1/n} = \lim_{n \to \infty} \frac{n}{\sqrt{n^2+1}} \cdot \frac{\frac{1}{n}}{1/n} = \lim_{n \to \infty} \frac{1}{\sqrt{1+1/n^2}} = 1.$$ Then since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ also diverges and so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ does <u>not</u> converge absolutely.

Conversely $\frac{1}{\sqrt{n^2+1}} \geq 0$ and $\frac{1}{\sqrt{(n+1)^2+1}} \leq \frac{1}{\sqrt{n^2+1}}$ so the sequence is decreasing and $\lim_{n \to \infty} \frac{1}{\sqrt{n^2+1}} = 0$. Thus, by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges & since it doesn't converge absolutely, it $\boxed{\text{converges conditionally}}$