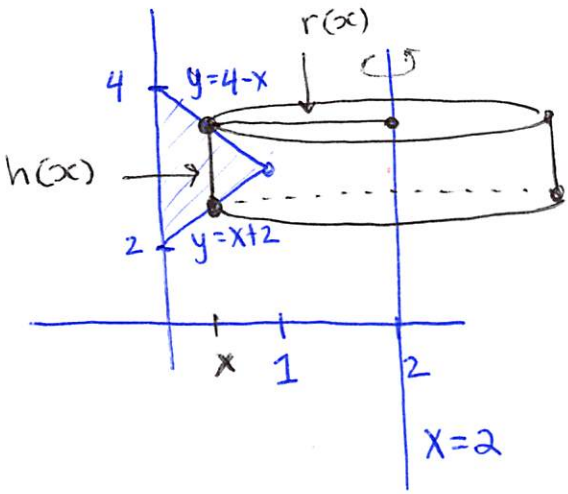
Practice Exam 2 solutions

#1 $y=4-x$, $y=x+2$, $x=0$ revolved around $x=2$



$4-x=x+2 \Rightarrow 2=2x \Rightarrow x=1$

Fix x.

$r(x)=2-x$

$h(x)=4-x-(x+2)=2-2x$

$$\text{vol}=2\pi\int_0^1 (2-x)(2-2x)\,dx = 4\pi\int_0^1 (2-x)(1-x)\,dx$$

$$=4\pi\int_0^1 2-x-2x+x^2\,dx = 4\pi\int_0^1 2-3x+x^2\,dx$$

F

$$=4\pi\left[2x-\frac{3x^2}{2}+\frac{x^3}{3}\right]_0^1 = 4\pi\left[2-3/2+1/3\right]=4\pi\cdot\frac{5}{6}=\frac{20\pi}{6}=\boxed{\frac{10\pi}{3}}$$

#2 ave value of $f(x)$ on $[0,2]$ $=\dfrac{1}{2-0}\displaystyle\int_0^2 \frac{2e^x}{1+e^{2x}}\,dx$ $u=e^x$ $du=e^x dx$

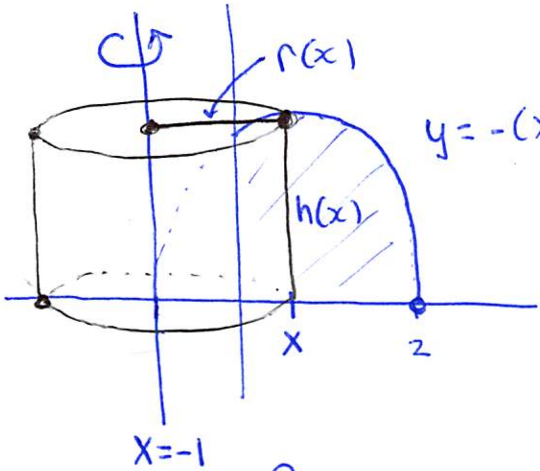
$$=\frac{1}{2}\int_1^{e^2}\frac{2\,du}{1+u^2}$$

$$=\arctan(u)\Big|_1^{e^2}$$

$$\boxed{=\arctan(e^2)-\frac{\pi}{4}}$$

C

3 $y = -(x-2)(x+1)$ $x=0$, $y=0$ revolved around $x=-1$.



$0 \le x \le 2$

Fix x

$r(x) = x+1$

$h(x) = -(x-2)(x+1)$

$$\text{Vol} = 2\pi \int_0^2 -(x-2)(x+1)^2\,dx = 2\pi \int_0^2 (2-x)[x^2+2x+1]\,dx$$

$$= 2\pi \int_0^2 2x^2+4x+2-x^3-2x^2-x\,dx$$

$$= 2\pi \int_0^2 -x^3+3x+2\,dx = 2\pi \left[-\frac{x^4}{4}+\frac{3}{2}x^2+2x\right]_0^2$$

$$= 2\pi[-4+6+4] = \boxed{12\pi}$$ (F)

---

4 $\int 3x^2[\ln x]^2 dx = x^3[\ln x]^2 - \int 2x^2 \ln x\,dx$

$u = [\ln x]^2$ $dv = 3x^2 dx$

$du = \frac{2}{x}\ln x$ $v = x^3$

$u = \ln x$ $dv = 2x^2 dx$

$du = 1/x\,dx$ $v = 2/3\,x^3$

$$= x^3[\ln x]^2 - \left[\frac{2}{3}x^3\ln x - \int 2/3\,x^2\,dx\right]$$

$$\boxed{= x^3[\ln x]^2 - 2/3\,x^3\ln x + 2/9\,x^3 + C}$$ (H)

#5 $\int \frac{\sin^3 x}{\cos^6 x}\, dx$

$= \int \frac{1}{\cos^3 x} \cdot \frac{\sin^3 x}{\cos^3 x}\, dx = \int \sec^3 x \cdot \tan^3 x \, dx$

$= \int \sec^2 x [\sec^2 x - 1] \cdot \sec x \tan x \, dx$

$u = \sec x \quad du = \sec x \tan x \, dx$

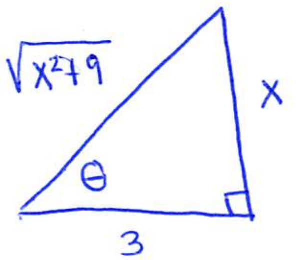
$= \int u^2 [u^2 - 1]\, du = \int u^4 - u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C$

$\boxed{= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$ C

#6 $\csc\left[\arctan\left(\frac{x}{3}\right)\right] = \csc\theta$

$\Rightarrow \theta = \arctan\left(\frac{x}{3}\right) \quad \Rightarrow x = 3\tan\theta$

$\Rightarrow \frac{x}{3} = \tan\theta = \frac{opp}{adj}$



This implies $\csc\theta = \frac{hyp}{opp} = \boxed{\frac{\sqrt{x^2+9}}{x}}$ C

#7 $\int_1^{\infty} \frac{2\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{2\ln x}{x^2}\,dx$ $\quad u=\ln x \quad dv = 2\frac{1}{x^2}dx$
$\quad du=\frac{1}{x}dx \quad V=-\frac{2}{x}$

$$= \lim_{t\to\infty}\left[-\frac{2\ln x}{x}\Big|_1^t + \int_1^t 2\frac{dx}{x^2}\right]$$

$$= \lim_{t\to\infty}\left[\frac{-2\ln t}{t} + \frac{2\ln 1}{1} - 2/x\Big|_1^t\right]$$

$$= \lim_{t\to\infty}\left[\overset{\infty/\infty}{\frac{-2\ln t}{t}}\right] - \lim_{t\to\infty}\frac{2}{t} + \lim_{t\to\infty} 2$$

$$= \lim_{t\to\infty} -\frac{2}{t} - \lim_{t\to\infty}\frac{2}{t} + 2 \quad = 0 + \boxed{2}$$

D

#8 $\int_{\pi/6}^{\pi/2} \cot^2 x\,dx = \int_{\pi/6}^{\pi/2} \csc^2 x - 1\,dx = -\cot x - x\Big|_{\frac{\pi}{6}}^{\pi/2}$

$$= -\cot\frac{\pi}{2} - \frac{\pi}{2} + \cot\frac{\pi}{6} + \frac{\pi}{6}$$

$\cos\frac{\pi}{2} = 0$
$\cos\pi/6 = \sqrt{3}/2$
$\sin\pi/6 = 1/2$

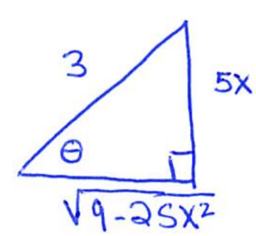
$$= \frac{-\cos\frac{\pi}{2}}{\sin\pi/2} - \frac{\pi}{2} + \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} + \frac{\pi}{6}$$

$$= 0 - \frac{\pi}{2} + \sqrt{3} + \frac{\pi}{6} \quad = \boxed{\sqrt{3} - \frac{\pi}{3}}$$

A

#9 $\int_0^{1/5} \frac{x^2}{9-25x^2}\,dx = \frac{1}{25}\int_0^{1/5}\frac{x^2}{9/25 - x^2}\,dx$ $\quad x = \frac{3}{5}\sin\theta \quad dx = \frac{3}{5}\cos\theta\,d\theta$
$\quad 9/25 - x^2 = 9/25\cos^2\theta$

$$\frac{1}{25}\int \frac{\frac{9}{25}\sin^2\theta\ \frac{3}{5}\cos\theta\,d\theta}{\frac{9}{25}\cos^2\theta} = \frac{3}{125}\int \frac{\overset{=1-\cos^2\theta}{\sin^2\theta}}{\cos\theta}\,d\theta$$



$$= \frac{3}{125}\int \sec\theta - \cos\theta\,d\theta = \frac{3}{125}\left(\ln|\sec\theta + \tan\theta| - \sin\theta\right) + C$$

Then: $\int_0^{.2}\frac{x^2}{9-25x^2}\,dx = \frac{3}{125}\ln\left|\frac{3}{\sqrt{9-25x^2}} + \frac{5x}{\sqrt{9-25x^2}}\right| - \frac{3}{125}\cdot\frac{5x}{3}\Big|_0^{.2}$

$\sec\theta = \frac{3}{\sqrt{9-25x^2}} \quad \tan\theta = \frac{5x}{\sqrt{9-25x^2}}$
$\sin\theta = 5x/3$

$$= \frac{3}{125}\ln\left|\frac{3}{\sqrt{8}} + \frac{1}{\sqrt{8}}\right| - 1/125$$

$$= 3/125\ln\sqrt{2} - 1/125$$

B

#10 $\displaystyle\int_0^1 \frac{2\,dx}{2x^2+3x+1}$

$2x^2+3x+1 = 2x^2+2x+x+1 = 2x(x+1)+(x+1) = (2x+1)(x+1)$

$\dfrac{2}{2x^2+3x+1} = \dfrac{A}{2x+1} + \dfrac{B}{x+1} \Rightarrow \quad 2 = A(x+1) + B(2x+1)$

$x=-1 \quad 2=-B \quad B=-2$

$x=-1/2 \quad 2=A(1/2) \quad A=4$

$$\int_0^1 \frac{2\,dx}{2x^2+3x+1} = \int_0^1 \frac{4}{2x+1} - \frac{2}{x+1}\,dx = 2\ln|2x+1| - 2\ln|x+1| \Big|_0^1$$

$$= 2\ln 3 - 2\ln 2 = \boxed{2\ln(3/2)}$$ F

#11 $x = \frac{2}{3}(y-1)^{3/2}$ between $1 \le y \le 4$

$x' = \frac{2}{3}\cdot\frac{3}{2}(y-1)^{1/2} = (y-1)^{1/2}$

$$\int_1^4 \sqrt{1+[x']^2}\,dy = \int_1^4 \sqrt{1+y-1}\,dy = \int_1^4 y^{1/2}\,dy$$

$$= \frac{2}{3}y^{3/2}\Big|_1^4 = \frac{2}{3}\left[4^{3/2}-1\right] = \frac{2}{3}[8-1] = \boxed{\frac{14}{3}}$$ E

#12 $\displaystyle\int_{-\infty}^0 \frac{dx}{\sqrt{3-x}} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{\sqrt{3-x}}$ $\quad u = 3-x$, $du = -dx$

$$= \lim_{t\to-\infty} \int_{3-t}^{3} -u^{-1/2}\,du$$

$$= \lim_{t\to-\infty} \int_3^{3-t} u^{-1/2}\,du$$

$$= \lim_{t\to-\infty} 2u^{1/2}\Big|_3^{3-t}$$

$$= \lim_{t\to-\infty} 2\sqrt{3-t} - 2\sqrt{3} \boxed{= \infty}$$ F

#13

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{3\cos\theta} = \int 9\sin^2\theta\, d\theta$$

$x = 3\sin\theta$ $\quad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta\, d\theta$

$$= \frac{9}{2}\int 1-\cos(2\theta)\,d\theta = \frac{9}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C$$

$$= \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C \qquad \sin\theta\cos\theta = \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{9}{2}\,\frac{x\sqrt{9-x^2}}{9} + C$$

$$\boxed{= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C}$$

C

#14

$$\int \sin(2t)\sin(6t)\,dt$$

use $\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

with $A = 2t$ and $B = 6t$.

Then $\sin(2t)\sin(6t) = \frac{1}{2}\left[\cos(2t-6t) - \cos(2t+6t)\right]$

$$= \frac{1}{2}\left[\cos(4t) - \cos(8t)\right]$$

so $\int \sin(2t)\sin(6t)\,dt = \frac{1}{2}\int \cos(4t) - \cos(8t)\,dt$

$$= \frac{1}{2}\left[\frac{1}{4}\sin(4t) - \frac{1}{8}\sin(8t)\right] + C$$

$$\boxed{= \frac{1}{8}\sin(4t) - \frac{1}{16}\sin(8t) + C}$$

A

#15 $\int_{-1}^{2} \frac{x}{(x+1)^2}\,dx$

$$\int_t^2 \frac{x}{(x+1)^2} = \int_t^2 \frac{x+1-1}{(x+1)^2}\,dx = \int_t^2 \frac{1}{x+1} - \frac{1}{(x+1)^2}\,dx$$

$$= \ln|x+1| + \frac{1}{x+1}\Big|_t^2$$

$$= \ln(3) + \frac{1}{3} - \left(\ln|t+1| + \frac{1}{t+1}\right)$$

$$\Rightarrow \int_{-1}^{2} \frac{x}{(x+1)^2}\,dx = \lim_{t\to -1^+} \int_t^2 \frac{x}{(x+1)^2}\,dx$$

$\infty - \infty$ b/c $\infty - \infty$ is unclear, we need to simplify

$$= \lim_{t\to -1^+} \ln(3) + \frac{1}{3} - \ln|t+1| - \frac{1}{t+1}$$

$$= \ln(3) + \frac{1}{3} - \lim_{t\to -1^+} \ln|t+1| + \ln\left[e^{\frac{1}{t+1}}\right]$$

$$= \ln(3) + \frac{1}{3} - \lim_{t\to -1^+} \ln\left|\frac{e^{\frac{1}{t+1}}}{\frac{1}{1+t}}\right|$$ ∞/∞

so we apply L'Hopital's Rule

$$= \ln(3) + \frac{1}{3} - \lim_{t\to -1^+} \ln\left|\frac{\frac{-1}{(t+1)^2} e^{1/t+1}}{\frac{-1}{(t+1)^2}}\right|$$

$$= \ln(3) + \frac{1}{3} - \lim_{t\to -1^+} \ln|e^{1/t+1}|$$

$\boxed{= -\infty}$

H

#16 $y=\frac{e^x}{4}+e^{-x}$ from $(0,\frac{5}{4})$ to $(1,\frac{e}{4}+e^{-1})$.

$$y'=\frac{e^x}{4}-e^{-x} \Rightarrow [y']^2=\left(\frac{e^x}{4}-e^{-x}\right)^2=\frac{e^{2x}}{16}-\frac{1}{2}+e^{-2x}.$$

Then $\sqrt{1+[y']^2}=\sqrt{1+[e^{2x}/16-1/2+e^{-2x}]}$

$$=\sqrt{\frac{e^{2x}}{16}+1/2+e^{-2x}}$$

$$=\sqrt{(e^x/4+e^{-x})^2}$$

$$=\frac{e^x}{4}+e^{-x}.$$

so arclength $=\int_0^1 \frac{e^x}{4}+e^{-x}\,dx=\frac{e^x}{4}-e^{-x}\Big|_0^1$

$$=\frac{e}{4}-e^{-1}-\left(\frac{1}{4}-1\right)=\boxed{\frac{e}{4}-e^{-1}+\frac{3}{4}}$$

E

---

17 $\int_0^{\pi/4} x\sec x\tan x\,dx$

$u=x \quad dv=\sec x\tan x\,dx$

$du=dx \quad v=\sec x$

$$=x\sec x\Big|_0^{\pi/4}-\int_0^{\pi/4}\sec x\,dx$$

$$=\frac{\pi}{4}\sec(\pi/4)-\ln|\sec x+\tan x|\Big|_0^{\pi/4}$$

$\sec\frac{\pi}{4}=\sqrt{2}$

$\tan\frac{\pi}{4}=1$

$$=\frac{\pi}{4}\cdot\sqrt{2}-\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|+\ln|\sec 0+\tan 0|$$

$$\boxed{=\frac{\pi\sqrt{2}}{4}-\ln(\sqrt{2}+1)}$$

C

#18 $\int \frac{x^2-x+6}{x^3+3x}\,dx$

$$\frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$\Rightarrow x^2-x+6 = A(x^2+3) + (Bx+C)x$$

$x=0$ $\quad 6 = 3A \Rightarrow \boxed{A=2}$

$$\Rightarrow x^2-x+6 = 2x^2+6+Bx^2+Cx$$

x^2: $1 = 2+B$ $\quad$ $\boxed{B=-1}$

x $\quad -1 = C$ $\quad$ $\boxed{C=-1}$

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \frac{2}{x} - \frac{1}{x^2+3} - \frac{x}{x^2+3}\,dx$$

$$= \int \frac{2}{x}\,dx - \int \frac{dx}{3\left[\left(\frac{x}{\sqrt{3}}\right)^2+1\right]} - \int \frac{x}{x^2+3}\,dx$$

$u = x/\sqrt{3}$ $\quad$ $w = x^2+3$

$du = 1/\sqrt{3}\,dx$ $\quad$ $dw = 2x\,dx$

$$= 2\ln|x| - \frac{1}{3}\int \frac{\sqrt{3}\,du}{u^2+1} - \int \frac{\frac{1}{2}\,dw}{w}$$

$$= 2\ln|x| - \frac{1}{\sqrt{3}}\arctan(u) - \tfrac{1}{2}\ln|w| + C$$

$$\boxed{= 2\ln|x| - \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) - \tfrac{1}{2}\ln|x^2+3| + C}$$

#19 $y = \sin x \quad 0 \le x \le \pi$

$y' = \cos x$

surface area $= \int_0^{\pi} 2\pi \sin x \sqrt{1+\cos^2 x}\, dx$ $\quad u = \cos x \quad x=0 \Rightarrow u=1$
$du = -\sin x\, dx \quad x = \pi \Rightarrow u = -1$

$$= \int_1^{-1} -2\pi \sqrt{1+u^2}\, du$$

$$= 2\pi \int_{-1}^{1} \sqrt{1+u^2}\, du$$ $\quad u = \tan\theta \quad -1 = \tan\theta \Rightarrow \theta = -\frac{\pi}{4}$
$du = \sec^2\theta\, d\theta \quad 1 = \tan\theta \Rightarrow \theta = \pi/4$

$$= 2\pi \int_{-\pi/4}^{\pi/4} \sec\theta \cdot \sec^2\theta\, d\theta$$

$u = \sec\theta \qquad dv = \sec^2\theta\, d\theta$
$du = \sec\theta\tan\theta\, d\theta \qquad v = \tan\theta$

Then: $\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta \cdot \underbrace{\tan^2\theta}_{\sec^2\theta - 1}\, d\theta$

$$= \sec\theta\tan\theta - \int \sec^3\theta + \int \sec\theta\, d\theta$$

so $\int \sec^3\theta\, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right] + C.$

Then:

surface area $= 2\pi \cdot \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_{-\pi/4}^{\pi/4}$

$$= \pi\left[\sqrt{2} + \ln(\sqrt{2}+1)\right] - \pi\left[-\sqrt{2} + \ln|\sqrt{2}-1|\right]$$

$$\boxed{= \pi\left[2\sqrt{2} + \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right]}$$