Final Exam: Sections 5.1-11.10

- Final exam: Fri Dec 16 10:30AM
- Rough final exam breakdown: Exam 1: 20%, Exam 2: 20%, Exam 3: 20%, Post Exam 3: 40%

- Reading Week:
  - Calculus Help Room Open
  - RPM Help
  - Thornton/Bickel Office Hours
  - Review Session (see emails)

1. Find \( \int_0^{\pi/4} \tan^3 x \sec^2 x \, dx \).
   Solution: \( u = \tan x \) gives answer 1/4.

2. Find \( \int xe^x \, dx \).
   Solution:

3. Find \( \int \frac{3}{x^2 + x - 2} \, dx \).
   Solution: \( \ln |x - 1| - \ln |x + 2| + C \)

4. Find the area between curves \( y = 2 - x^2 \) and \( y = 2 - 2x \).
   Solution: 4/3

5. Find the volume obtained by rotating the region between the \( y \) axis and the curve \( x = \frac{2}{y} \) for \( 1 \leq y \leq 4 \), about the \( y \)-axis.
   Solution: \( V = \pi \int_1^4 \frac{4}{y^2} \, dy = 3\pi \)

6. Find \( \int_0^\infty \frac{x}{(x^2 + 2)^2} \, dx \)
   Solution: 1/4

7. Find the sum \( \sum_{n=1}^\infty \frac{2n-1}{3^n} \)
   Solution: 3/2

8. Find the interval of convergence of \( \sum_{n=1}^\infty \frac{2^n(x-4)^n}{n3^n} \)
   Solution: \([5/2, 11/2])

9. The Maclaurin series for \( x \cos(x/2) = \sum_{n=0}^\infty a_n x^n \). Find \( a_5 \).
   Solution: 1/384.

10. If \( \int \arctan(x^2) \, dx = C + \sum_{n=1}^\infty a_n x^n \), find \( a_7 \).
    Solution: -1/21.

11. If \( \sin x = \sum_{n=0}^\infty a_n (x - \pi/2)^n \). Find \( a_3 \).
    Solution: -1/12.

12. How many terms of the series are needed to approximate \( \int_0^1 \cos(x^3) \, dx \) to an error less than \( 10^{-4} \)? What is this estimate?
    Solution: 3 terms needed: \( 1 - 1/14 + 1/312 = .931777 \)

13. Find \( \int_1^e x^2 \ln x \, dx \)
    Solution: \( (2e^3 + 1)/9 \)

14. Find \( \int \frac{1}{x^2 + 2} \, dx \)
    Solution: \( \ln |x/(x - 1)| + C \)

15. Find the length of \( x = (2/3)y^{3/2} \), \( 0 \leq y \leq 3 \).
    Solution: 14/3.

16. Find \( T_2 \), centered at \( x = 1 \) for \( f(x) = 1/x \).
    Solution: \( T_2(x) = 1 - (x - 1) + (x - 1)^2 \).
17. If \( xe^{2x} = \sum_{n=1}^{\infty} b_n x^n \), find \( b_4 \).
   Solution: \( 4/3 \).

18. Find Maclaurin series for \( x^2 - x \) \( \arctan x \)
   Solution: \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{2n+3} \).

19. Find \( \int_0^8 \frac{1}{\sqrt[3]{x^2}} \) \( dx \)
   Solution: \( 6 \).

20. Find the sum \( 3 - \frac{6}{2} + \frac{12}{25} - \frac{24}{125} + \frac{48}{625} - \cdots \)
   Solution: \( 15/7 \).

21. Find a function represented by Maclaurin series:
   \( 1 + 3x + 9x^2 + 27x^3 + 81x^4 + \cdots \)
   Solution: \( 1/(1 - 3x) \).

22. Find the interval of convergence for \( \sum_{n=0}^{\infty} \frac{(3x-2)^n}{2^n} \)
   Solution: \( (0, 4/3) \).

23. Find \( \int_1^2 (2x - 1) \cos(x^2 - x) \) \( dx \)
   Solution: sin 2

24. Find the radius of convergence of \( \sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n!} \)
   Solution: 4.

25. If \( \frac{1}{e^x} = \sum_{n=0}^{\infty} d_n (x - 2)^n \) find \( d_3 \).
   Solution: \(-24/192 \).

26. Find \( f(x) \) and \( b \) if: \( 3 + \int_0^x \frac{f(t)}{t^4} dt = 24x^{-3} \).
   Solution: Plugging in \( x = b \) makes the integral equal to 0 and gives the equation \( 3 + 0 = 24b^{-3} \). Solving gives \( b = 2 \).
   Take the derivative of both sides to find \( f(x): \frac{f(x)}{x^4} = -72x^{-4} \), which gives \( f(x) = -72 \).

27. Let \( f(x) = \int_x^2 \frac{5+t^4}{\sqrt{1+t^2}} \) \( dt \). Find \( f'(2) \).
   Solution: \(-7 \).

28. Determine convergence or divergence of \( \sum_{n=1}^{\infty} (-1)^n \left( \frac{n}{n+1} \right)^n \)
   Solution: Diverges since \( \lim_{n \to \infty} a_n \neq 0 \).

29. Determine convergence or divergence of \( \sum_{n=0}^{\infty} \frac{(-1)^n n^{2/3}}{1+n^n} \).
   Solution: Converges conditionally but not absolutely.

30. Determine convergence or divergence of \( \sum_{n=1}^{\infty} \frac{(3+n)^n}{3n} \)
   Solution: Converges absolutely by root test.

31. Let \( f(x) = \frac{1}{\sqrt{2}} x^3 e^{2x^2} \). Find \( f^{(7)}(0) \).
   Solution: Write out the Maclaurin series and look at the 7th term to get answer: 140.

32. Find the interval of convergence for \( \sum_{n=0}^{\infty} \frac{(-1)^n (x+3)^n}{4^n \sqrt{5+n}} \)
   Solution: Radius of convergence is 4. Interval: \((-7, 1)\)

33. Let \( T_2 \) be the degree 2 taylor polynomial centered at 2 for \( \ln x \). Find \( T_2(3) - \ln 2 \).

34. Find the coefficient of \( x^5 \) in the Maclaurin series for \( \sum_{n=0}^{\infty} \frac{8x}{4-x^2} \).
   Solution: 1/8

35. Find the coefficient of \( x^4 \) in the Maclaurin series for \( \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^{1/3}} \).
   Solution: 2/9

36. Find the radius and interval of convergence for \( \sum_{n=1}^{\infty} \frac{n}{b^n} (x - 1)^n \) where \( b > 0 \).
   Solution: \( R = b \) and \( I = (1 - b, 1 + b) \).

37. Find all \( p \) such that \( \int_2^\infty \frac{(\ln x)^p}{x} \) \( dx \) converges.
   Solution: \( p < -1 \).
38. Find the area between \( y = \frac{10}{x} \) and \( y = 7 - x \).

**Solution:** Area is \( \left[ 7x - \frac{x^2}{2} - 10 \ln x \right]_2^5 \) (whatever that equals).

39. Find the sum of \( 1 - e + e^2/2! - e^3/3! + e^4/4! - \cdots \)

**Solution:** \( 1/e \).

40. If \( g(x) = \int_2^3 \sqrt{1+t^3} \, dt \) then what is \( g'(1) \)?

**Solution:** \( -2\sqrt{2} \).

41. (T/F) If \( r > 1 \) and \( a \neq 0 \) then \( \sum_{n=1}^{\infty} ar^n \) converges.

**Solution:** False.

42. (T/F) If \( \sum_{n=1}^{\infty} c_n (-2)^n \) converges then \( \sum_{n=1}^{\infty} \frac{c_n}{2^n} \) converges.

**Solution:** True.

43. (T/F) If \( \sum_{n=1}^{\infty} c_n (x - 10)^n \) converges for all \( x \) then \( \sum_{n=1}^{\infty} c_n n (x - 10)^{n-1} \) converges for all \( x \). Converges.

**Solution:** True.

44. (T/F) If \( \sum_{n=1}^{\infty} a_n \) converges then \( \sum_{n=1}^{\infty} (-1)^n a_n \) converges.

**Solution:** False: maybe it does maybe it doesn’t (see if you can find examples of both).

45. (T/F) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2\pi}{n} \sin \left( \frac{2\pi i}{n} \right) = 0 \)

**Solution:** True. Write it as a Riemann sum.

46. Approximate \( \int_{0}^{3} x^2 \, dx \) by using the midpoint rule and 3 subdivisions.

**Solution:** 35/4.

47. Let \( g(x) = \int_{1}^{x} \arctan t \, dt \). Find \( g'(x) \).

**Solution:** \( (-1/x^2) \arctan(1/x) \).

48. Find \( \int \frac{x + 1}{x^2 + 1} \, dx \)

**Solution:** \( \frac{1}{2} \ln |x^2 + 1| + \arctan x + C \)

49. Find the volume of the solid obtained by rotating about the \( y \)-axis the region bounded by \( y = x^3 \), \( y = 8 \) and \( x = 0 \).

**Solution:** 96\pi/5.

50. Given that \( \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \cdots \). How many terms must be added to obtain a partial sum within \( \frac{1}{100} \) of \( \ln 2 \)?

**Solution:** If you add 99 terms then the next term is \(-1/100\) and thus the maximum error is \( 1/100 \). Thus, you need 99 terms.