12-9 Review Packet Solutions

#1 a) $\sum_{n=0}^{\infty} \frac{n^2}{(2n+1)!}$ Use Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2(n+1)+1)!}\cdot\frac{(2n+1)!}{n^2} = \lim_{n\to\infty}\frac{1}{(2n+2)(2n+3)}\cdot\frac{n^2+2n+1}{n^2} = 0 < 1$$

$\therefore$ series converges by Ratio Test

b) $\sum_{n=0}^{\infty} \frac{2^n n!}{n^n}$ Use Ratio Test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n\cdot n!} = \lim_{n\to\infty} 2\cdot\left[\frac{n}{n+1}\right]^n = L$$

$$\Rightarrow \ln(L) = \lim_{n\to\infty} \ln 2 + n\ln\left(\frac{n}{n+1}\right) = \ln 2 + \lim_{n\to\infty}\frac{-\ln[1+1/n]}{1/n}\ \ 0/0$$

$$= \ln 2 + \lim_{x\to\infty}\frac{+1/x^2 / 1+1/x}{-1/x^2} = \ln 2 - 1 = \ln 2 - \ln e = \ln[2/e]$$

$\therefore$ limit $L = 2/e < 1$ so converges by Ratio Test

c) $\sum_{n=0}^{\infty}\left[\arctan(1/n)\right]^n$ use Root Test

$$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\arctan(1/n) = \arctan(0) = 0 < 1$$

$\therefore$ series converges by the Root Test

#2 a) $\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^3}x^n$ Use Ratio Test

$$\lim_{n\to\infty}\left|\frac{(2(n+1))!\,x^{n+1}}{((n+1)!)^3}\cdot\frac{(n!)^3}{(2n)!\,x^n}\right| = \lim_{n\to\infty}\left|x\cdot\frac{(2n+2)!}{(2n)!}\cdot\left[\frac{n!}{(n+1)!}\right]^3\right|$$

$$= \lim_{n\to\infty}|x|\cdot\frac{(2n+1)(2n+2)}{(n+1)(n+1)(n+1)} = 0.$$ $\therefore$ converges for all x

Interval of convergence is $(-\infty, \infty)$

b) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$ Use Ratio Test

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right| = \lim_{n\to\infty} |x-2| \cdot \left(\frac{n^2+1}{(n+1)^2+1} \right) = |x-2| < 1$$

$\Rightarrow$ converge for $-1 < x-2 < 1 \Rightarrow 1 < x < 3$.

check endpoints:

$x=1$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ note $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges by Direct Comparison b/c

$\frac{1}{n^2+1} \le \frac{1}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges b/c p-series w/ $p=2$

$\therefore$ original series converges b/c converges absolutely.

$x=3$ $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges by above argument.

$\therefore$ interval of convergence $= [1,3]$

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c) $\sum_{n=0}^{\infty} 3^n (x+3)^{2n}$ Use Ratio Test

$$\lim_{n\to\infty} \left| \frac{3^{n+1}(x+3)^{2(n+1)}}{3^n (x+3)^{2n}} \right| = \lim_{n\to\infty} 3|x+3|^2 = 3|x+3|^2 < 1 \Rightarrow |x+3| < 1/\sqrt{3}$$

so $-1/\sqrt{3} < x+3 < 1/\sqrt{3}$ so conv if $-3-1/\sqrt{3} < x < 1/\sqrt{3} - 3$

check endpoints: $x = -3 - 1/\sqrt{3}$

$$\sum_{n=0}^{\infty} 3^n (-3 - 1/\sqrt{3} + 3)^{2n} = \sum_{n=0}^{\infty} 3^n (1/3)^n = \sum_{n=0}^{\infty} (-1)^n \text{ diverges}$$

$x = 1/\sqrt{3} - 3$ $\sum_{n=0}^{\infty} 3^n (1/\sqrt{3} - 3 + 3)^{2n} = \sum_{n=0}^{\infty} 3^n (1/3)^n = \sum_{n=0}^{\infty} 1$ diverges

$\therefore$ Interval of convergence $= (-3-1/\sqrt{3}, -3+1/\sqrt{3})$

#3 a) $\frac{2x^2}{(1-4x)^2}$

note: $\frac{1}{1-4x} = \sum_{n=0}^{\infty} 4^n x^n \Rightarrow$ differentiating $\frac{4}{(1-4x)^2} = \sum_{n=1}^{\infty} 4^n \cdot n \cdot x^{n-1}$ for $|4x|<1 \Rightarrow |x|<1/4$

$$\therefore \frac{2x^2}{(1-4x)^2} = \frac{2x^2}{4} \cdot \frac{4}{(1-4x)^2} = \frac{x^2}{2} \sum_{n=1}^{\infty} 4^n n \cdot x^{n-1} = \sum_{n=1}^{\infty} \frac{4^n}{2} \cdot n \cdot x^{n+1} \quad R = 1/4$$

b) $\ln(2x+3) = \int \frac{2}{2x+3}\,dx = \int \frac{2/3}{1--2x/3}\,dx = \sum_{n=0}^{\infty} 2/3 \int (-2x/3)^n\,dx$ $\quad |2x/3|<1 \Rightarrow |x|<3/2$

$$= \sum_{n=0}^{\infty} \frac{2/3 \cdot (2/3)^n (-1)^n x^{n+1}}{n+1} + C$$

$x=0 \Rightarrow \ln 3 = C$

$$\therefore \ln(2x+3) = \sum_{n=0}^{\infty} \frac{(-1)^n (2/3)^{n+1} x^{n+1}}{n+1} + \ln 3 \qquad R=3/2$$

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d) $x\sin(2x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+2}}{(2n+1)!}$ $\qquad R=\infty$

#4 a)

$f(x) = x^{1/3}$ $\quad f(8)=2$ $\quad a_0 = 2$

$f'(x) = 1/3\, x^{-2/3}$ $\quad f'(8) = \frac{1}{3}\cdot\frac{1}{4}$ $\quad a_1 = 1/12$

$f''(x) = -2/9\, x^{-5/3}$ $\quad f''(8) = -\frac{2}{9}\cdot\frac{1}{32}$ $\quad a_2 = -\frac{1}{9\cdot 16}\cdot\frac{1}{2!}$

$f'''(x) = \frac{10}{27} x^{-8/3}$ $\quad f'''(8) = \frac{10}{27}\cdot\frac{1}{256} = \frac{5}{27\cdot 128}$ $\quad a_3 = \frac{5}{27\cdot 128}\cdot\frac{1}{3!}$

$$2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20{,}736}(x-8)^3$$

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b)

$f(x) = \sin x$ $\quad f(\pi/6) = 1/2$ $\quad a_0 = 1/2$

$f'(x) = \cos x$ $\quad f'(\pi/6) = \sqrt{3}/2$ $\quad a_1 = \sqrt{3}/2$

$f''(x) = -\sin x$ $\quad f''(\pi/6) = -1/2$ $\quad a_2 = -1/4$

$f^{(3)}(x) = -\cos x$ $\quad f^{(3)}(\pi/6) = -\sqrt{3}/2$ $\quad a_3 = -\sqrt{3}/12$

$$1/2 + \sqrt{3}/2\,(x-\pi/6) - 1/4\,(x-\pi/6)^2 - \sqrt{3}/12\,(x-\pi/6)^3$$

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c) $f(x) = e^x \cos x$

$$= \left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right]\left[1 - x^2/2 + x^4/24 \pm \ldots\right]$$

$$= 1 + x - x^2/2 + x^2/2 - x^3/2 - x^4/4 + \frac{x^4}{24} + \frac{x^4}{24} \pm \ldots$$

$$= \underbrace{1 + x - x^3/2 - 1/6\, x^4}_{\text{1st 4 nonzero terms}} \pm \ldots$$

#5 a) $f(x) = \ln x \Rightarrow f(2) = \ln 2$

$f'(x) = 1/x$

$f''(x) = -1/x^2$

$f^{(3)}(x) = 2x^{-3}$

$f^{(4)}(x) = 2 \cdot (-3) x^{-4}$

$\vdots$

$n \geq 1 \quad f^{(n)}(x) = (-1)^{n+1} (n-1)!\, x^{-n}$

$$\frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n+1}(n-1)!\,2^{-n}}{n!} = \frac{(-1)^{n+1}}{n 2^n}$$

$$\therefore \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, 2^n} x^n + \ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)\, 2^{n+1}} + \ln 2$$

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b) $f(x) = \sin(2x)$ centered at π

$$f(x) = \sin(2(x-\pi+\pi)) = \sin(2(x-\pi)+2\pi) = \sin[2(x-\pi)]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n [2(x-\pi)]^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} (x-\pi)^{2n+1}}{(2n+1)!}$$

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c) $f(x) = x^2 \ln(1+x^3) \quad c=0$

$$= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+4}}{n+1}$$

#6 a) $$\int \sin(x^3)\,dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{6n+3}}{(2n+1)!} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+4}}{(6n+4)(2n+1)!}$$

$$= C + \frac{x^4}{4} - \frac{x^{10}}{10 \cdot 3!} \pm \ldots$$

$\Rightarrow a_1 = 0$ & $a_4 = 1/4$ so $a_1 + a_4 = 1/4$

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b) $$\int \frac{\cos x - 1}{x} dx = \int \frac{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} - 1}{x} dx = \int \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n)!} dx$$

$$= C + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n \cdot (2n)!} = C - \underbrace{\frac{x^2}{2 \cdot 2}}_{n=1} + \underbrace{\frac{x^4}{4 \cdot 24}}_{n=2} \pm \ldots$$

$\Rightarrow a_1 = 0 \quad a_4 = \frac{1}{96}$ so $a_1 + a_4 = \frac{1}{96}$

d) $\int \ln(x^2+1)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+3)(n+1)}$

$= C + \underset{n=0}{\frac{x^3}{3}} - \underset{n=1}{\frac{x^5}{5\cdot 2}} \pm \dots$

$\Rightarrow a_1 = 0$ & $a_4 = 0$ so $a_1 + a_4 = 0$

#7 a) $f(x) = (x+1)^{1/2}$ $c=0$ $a_0 = 1$

$f'(x) = 1/2\,(x+1)^{-1/2}$ $a_1 = 1/2$

$f''(x) = -1/4\,(x+1)^{-3/2}$ $a_2 = -1/4 \cdot 1/2 = -1/8$

$f'''(x) = 3/8\,(x+1)^{-5/2}$ $a_3 = 3/8 \cdot 1/6 = 3/48$

$f^{(4)}(x) = -15/16\,(x+1)^{-7/2}$ $a_4 = -15/16 \cdot 1/24 = -5/128$

$\therefore T_4(x) = 1 + 1/2\,x - 1/8\,x^2 + 3/48\,x^3 - 5/128\,x^4$

$\Rightarrow f(1/2) \approx T_4(1/2) = 1 + 1/4 - 1/8 \cdot 1/4 + 3/48 \cdot 1/8 - 5/128 \cdot 1/16$

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b) $f(x) = \sin x$ $c=0$

we know $T_5(x) = x - x^3/3! + x^5/5!$

so $\sin(1) \approx T_5(1) = 1 - 1/6 + 1/120$

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c) $f(x) = e^x$ $c=1$

$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(1) = e \Rightarrow a_n = \frac{e}{n!}$

$\Rightarrow T_4(x) = e + e(x-1) + e/2\,(x-1)^2 + e/3!\,(x-1)^3 + e/4!\,(x-1)^4$

$f(2) = e^2 \approx T_4(2) = e\left[1 + 1 + 1/2 + 1/3! + 1/4!\right]$

#8 a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{n+1}}{n+1} - 1/2$

$= \ln(1/2) - 1/2$

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b) $\sum_{n=0}^{\infty} \frac{(n+1)}{3^n} = \sum_{n=0}^{\infty} (n+1)\cdot(1/3)^n = \sum_{n=1}^{\infty} n\cdot(1/3)^{n-1} = \frac{1}{(1-1/3)^2} = \frac{1}{(2/3)^2}$

$= 9/4$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n}$

$\lim_{n\to\infty} \frac{(-1)^n n!}{2^n}$ $\to \infty$ n even, $\to -\infty$ n odd $\therefore$ series diverges by n^{th} term Test For Divergence