Solution

Dec. 7th: 11.10 : Taylor and Maclaurin Series & 11.11 Applications

Warm-up: Fill in the following table from memory:

Function	Maclaurin Series/ Power Series Rep centered at 0
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$
$\frac{1}{(1-x)^2}$	$\sum_{n=1}^{\infty} nx^{n-1}$
$\ln(x+1)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$
$\arctan(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$

In-Class Exercises

1. Find the value of the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{2^n}{n!} \Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n!} = e^2 - 1$$

2. (Clicker) Find the value of the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(\frac{\pi}{6})^{2n}}{2(2n)!}$.

$$= \frac{-1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/6)^{2n}}{(2n)!} = -\tfrac{1}{2}\cos(\pi/6) = -\tfrac{1}{2}\cdot \tfrac{\sqrt{3}}{2} = -\sqrt{3}/4$$

a. $\frac{1}{2}$

b. $-\frac{1}{4}$

c. $\frac{\sqrt{3}}{2}$

d. $-\frac{\sqrt{3}}{4}$

3. a. Evaluate the indefinite integral $\int e^{-x^2}\, dx$ as a power series centered at 0.

$$\int e^{-x^2} dx = \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + C$$

b. If we write $\int e^{-x^2}\, dx = C + \sum_{n=1}^{\infty} a_n x^n$, what is $a_1 + a_2 + a_3$?

by (a)

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2} \pm \dots \Rightarrow a_1 = 1, \quad a_2 = 0, \quad a_3 = -1/3 \quad \text{so } a_1 + a_2 + a_3 = 2/3$$

c. Estimate the definite integral $\int_0^1 e^{-x^2}\, dx$ to an error of at most .001.

$$\int_0^1 e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} := S$$

By A.S. Est. theorem $|S - S_N| \le b_{N+1} = \frac{1}{(2N+1)\cdot N!} \le .001$ if $N = 4$

$S \approx S_4 \approx .747$

5. Use series to evaluate $\frac{d^3}{dx^3}(\arctan(x))\,|_{x=0}$.

$$\frac{d}{dx}[\arctan x] = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)x^{2n}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow \frac{d^2}{dx^2}[\arctan x] = \sum_{n=1}^{\infty} (-1)^n (2n) x^{2n-1}$$

$$\frac{d^3}{dx^3}[\arctan x] = \sum_{n=1}^{\infty} (-1)^n (2n)(2n-1) x^{2n-2} \Rightarrow \frac{d^3}{dx^3}[\arctan x]\Big|_0 = -2\cdot(2-1) = -2$$

4. Use series to evaluate the following limit: $\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}$

$$\lim_{x\to 0} \frac{1 - [1 - x^2/2 + x^4/4! \pm \dots]}{1 + x - [1 + x + x^2/2 + x^3/3! + \dots]} = \lim_{x\to 0} \frac{x^2/2 - x^4/4! \pm \dots}{-x^2/2 - x^3/3! - x^4/4! - \dots} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to 0} \frac{1/2 - x^2/4! \pm \dots}{-1/2 - x/3! - x^2/4! - \dots}$$

$$= -1$$

6. a. Approximate $f(x) = \sin x$ with a Taylor polynomial of degree 5 centered at 0.

$$f(x) \approx T_5(x) = x - x^3/3! + x^5/5!$$

b. Use Taylor's Inequality to estimate the accuracy of the approximation when $-2 < x < 2$.

$|f^{(5+1)}(x)| \le 1$ so

$$|f(x) - T_5(x)| \le \frac{1\cdot |x|^5}{6!} \le \frac{32}{720}$$