Solutions

Dec. 5th: 11.10 : Taylor and Maclaurin Series

Warm-up

Fill in as much of the following table as you can:

Function	Maclaurin Series	Radius of Convergence
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1+x+x^2+\dots$	$R=1$
$\frac{1}{(1-x)^2}$	$\sum_{n=1}^{\infty} n\cdot x^{n-1} = 1+2x+3x^2+\dots$	$R=1$
$\ln(x+1)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x-\frac{x^2}{2}+\frac{x^3}{3}+\dots$	$R=1$
$\arctan(x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}+\dots$	$R=1$
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\dots$	$R=\infty$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\dots$	$R=\infty$

use Ratio Test

In-Class Exercises

Taylor's Inequality. If $|f^{(N+1)}(x)| \leq M$ for $|x-c| \leq r$, then the remainder $R_N(x)$ of the Taylor Series satisfies the inequality

$$|f(x) - T_N(x)| = |R_N(x)| \leq \frac{M}{(N+1)!}|x-c|^{N+1}$$

for all $|x-c| \leq r$.

1. Show that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

Fix $r>0$ and consider x in $(-r,r)$. Then $|f^{(N+1)}(x)| = e^x \leq e^r$. By Taylor's Inequality

$$0 \leq |R_N(x)| \leq \frac{e^r}{(N+1)!}|x|^{N+1} \to 0 \text{ as } N \to \infty.$$

By the squeeze theorem $|R_N(x)| \to 0$ so $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ For all x in $(-r,r)$. Since every x is in some $(-r,r)$ interval,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ For all } x.$$

2. Show that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ for all x.

Fix $r>0$ and consider x in $(-r,r)$. Then $|f^{(N+1)}(x)| = |\sin x|$ or $|\cos x| \leq 1$ and so By Taylor's Inequality,

$$0 \leq |R_N(x)| \leq \frac{|x|^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty.$$

By the Squeeze Thm, $|R_N(x)| \to 0$ so $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ For x in $(-r,r)$.
Since every x is in some $(-r,r)$; we have:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{For all } x.$$

3. Find the Maclaurin series for $\cos x$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \cos x = \sum_{n=0}^{\infty} \frac{d}{dx}\left[\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)\cdot(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$$

4. (Clicker) Find the Taylor series for $\ln(2x^2+1)$ with center 0.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{2n+2}}{n+1}$

b. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{2n+2}}{2n+1}$

c. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^{2n+1}}{n+1}$

d. $\sum_{n=0}^{\infty} \frac{2^{n+1} x^{2n+1}}{n+1}$

$$\ln(x+1) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

$$\Rightarrow \ln(2x^2+1) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{2n+2}}{n+1}$$

5. Find the first ~~four~~ three nonzero terms of the Maclaurin series for $e^x \sin x$.

$$e^x \cdot \sin x = \left[1 + x + x^2/2 + x^3/6 + \ldots\right]\left[x - x^3/6 + x^5/120 \pm \ldots\right]$$

$$= x + x^2 + x^3/2 - x^3/6 - x^4/6 + x^4/6 + \ldots$$

$$= x + x^2 + x^3/3$$