Math 132 - December 2, 2016

Solutions

Section 11.10: Taylor Series

- Taylor Series
  $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ where $c_n = \dfrac{f^{(n)}(a)}{n!}$.
- $T_n(x)$ is the Taylor Polynomial of Degree n.

Warm-up Problems

1. $\boxed{\textbf{Clicker}}$ Compute $\displaystyle\int_0^{1/2} \cos(x^2)\, dx$
   (a) 0 (b) 0.5 (c) 1 (d) $\pi/2$ (e) Who knows.
   Solution: This is impossible to find a useful anti-derivative using standard techniques. Instead, see Problem 6.
2. Find the Taylor Series for $f(x) = \cos x$ centered at $x = 0$
   Solution:
   $$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots$$

Class Problems

Lecture Notes: Like yesterday, the goal is:

$\boxed{\text{Given a function } f(x)\text{, find a representation of } f(x) \text{ with power series.}}$

We want to expand our skills to center the power series we find to somewhere other than $x = 0$.
Idea of attack for a starting function $f(x)$.

- Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$
- Plug in $x = a$ to get $c_0 = f(a)$
- $\frac{d}{dx}$ gives $f'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1}$
- Plug in $x = a$ to get $c_1 = f'(a)$
- $\frac{d}{dx}$ gives $f''(x) = \sum_{n=2}^{\infty} c_n n(n-1)(x-a)^{n-2}$
- Plug in $x = a$ to get $c_2 = f''(a)/2$
- Continue to get $c_n = \frac{f^{(n)}(a)}{n!}$

3. Here we find the Taylor Series for $f(x) = \ln x$ centered at $x = 1$.

(a) Fill in the chart below

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$c_n = \frac{f^{(n)}(1)}{n!}$
0	$\ln x$	0	0
1	$\frac{1}{x}$	1	1
2	$-\frac{1}{x^2}$	-1	$-\frac{1}{2}$
3	$\frac{2}{x^3}$	2	$\frac{1}{3}$
4	$-\frac{3!}{x^4}$	$-3!$	$-\frac{1}{4}$
5	$\frac{4!}{x^5}$	$4!$	$\frac{1}{5}$
6	$-\frac{5!}{x^6}$	$-5!$	$-\frac{1}{6}$
n	$\frac{(-1)^{n-1}(n-1)!}{x^n}$	$(-1)^{n-1}(n-1)!$	$\frac{(-1)^{n-1}}{6}$

(b) Write down the power series (fill in the c_n):

$$\begin{aligned}\ln x =& c_0 + c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^3 + c_4(x-1)^4 + c_5(x-1)^5 + c_6(x-1)^6 + \cdots \\ =& (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + \cdots \\ =& \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}\end{aligned}$$

4. **Clicker** Given a function, $f(x)$, it is always possible to find a Taylor Series that equals $f(x)$.
(a) True (b) False **Correct**

Solution: But, the answer is actually "mostly true".

Here are some example that cause problems

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-1/x^2} & \text{if } x \neq 0 \end{cases}$$

Note that $g(x)$ is continuous and infinitely differentiable but the Taylor Series for $g(x)$ is just 0 (kinda tricky to show this).

5. **Clicker** Suppose $f(x)$ is a function and $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is its Taylor Series.
Sometimes it is easier to make computations with a Taylor Polynomial of f rather than $f(x)$.
(a) True **Correct** (b) False

Important Series to Know (or at least be familiar with):

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$(1+x)^k =$(Its in the text but I don't think its important enough to fuss with.)

6. Use Taylor Series to compute $\int_0^{1/2} \cos(x^2)\, dx$

   (a) Find a series for

   $$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \cdots$$

   (b) Integrate your series

   $$\int \cos(x^2) \; = x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \frac{x^{17}}{17 \cdot 8!} - \cdots$$

   (c) Integrate your series

   $$\int_0^{1/2} \cos(x^2) \; = \frac{1}{2} - \frac{1}{5 \cdot 2^5 \cdot 2!} + \frac{1}{9 \cdot 2^9 \cdot 4!} - \frac{1}{13 \cdot 2^{13} \cdot 6!} + \frac{1}{17 \cdot 2^{17} \cdot 8!} - \cdots$$

   (d) Approximate your series by taking a couple terms. How accurate are you?
   Solution:

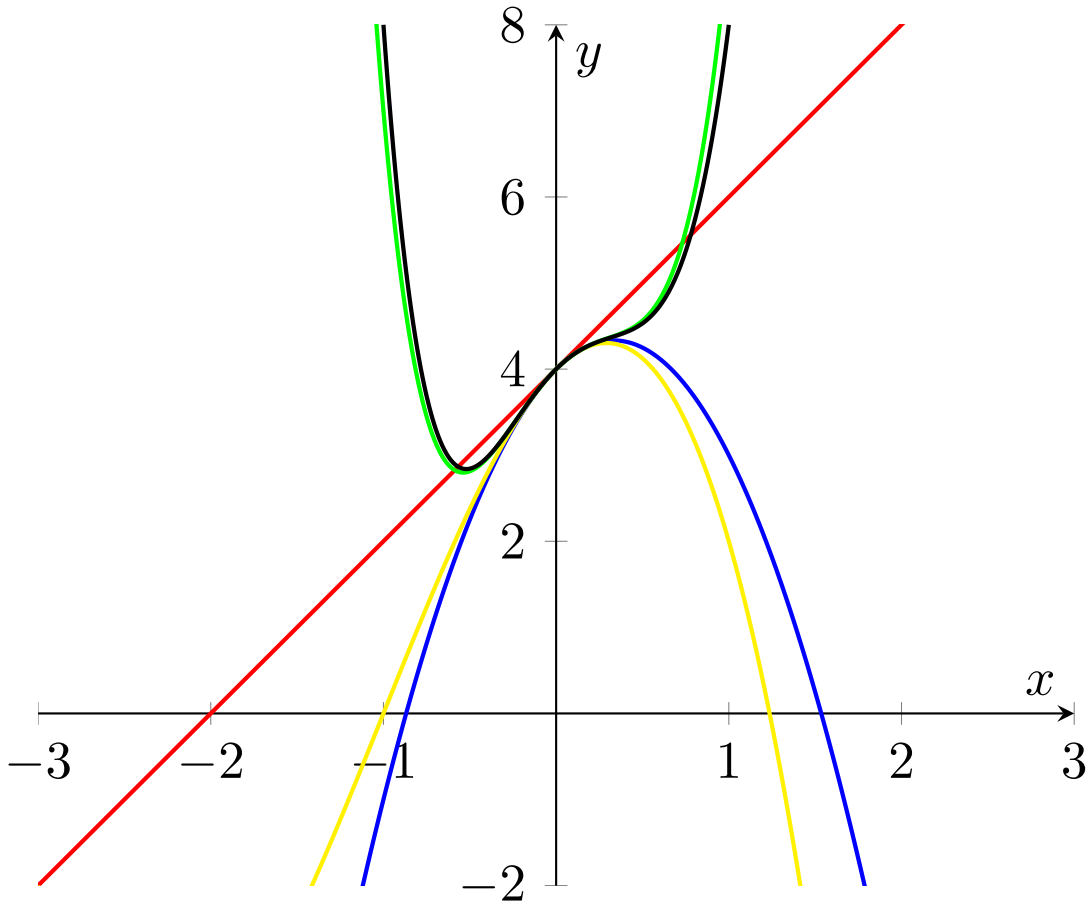
n	S_n	Max Error
1	0.5	0.003125
2	0.496875	0.00000904
3	0.49688404	1.3×10^{-8}
4	0.49688403	1.1×10^{-11}

7. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 0$.
   Solution:

$$4 + 2\,x - 3\,x^2 - x^3 + 7\,x^4 - x^5$$

Lets graph some of these Taylor Polynomials.

T_1: Red
T_2: Blue
T_3: Yellow
T_4: Green
$T_5 = f$: Black



8. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 1$.

Solution:

$$8 + 16(x-1) + 26(x-1)^2 + 17(x-1)^3 + 2(x-1)^4 - (x-1)^5$$

Lets graph some of these Taylor Polynomials.

T_1: Red
T_2: Blue
T_3: Yellow
T_4: Green
$T_5 = f$: Black