## Section 11.10: Taylor Series

- **Taylor Series**
  \[
  f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(a)}{n!}.
  \]
- **T_n(x)** is the Taylor Polynomial of Degree n.

### Warm-up Problems

1. **Clicker** Compute \( \int_0^{1/2} \cos(x^2) \, dx \)
   
   (a) 0  (b) 0.5  (c) 1  (d) \(\pi/2\)  (e) Who knows.

   **Solution:** This is impossible to find a useful anti-derivative using standard techniques. Instead, see Problem 6.

2. Find the Taylor Series for \( f(x) = \cos x \) centered at \( x = 0 \)

   **Solution:**
   
   \[
   1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots
   \]

### Class Problems

**Lecture Notes:** Like yesterday, the goal is:

**Given a function** \( f(x) \), find a representation of \( f(x) \) with power series.

We want to expand our skills to center the power series we find to somewhere other than \( x = 0 \).

Idea of attack for a starting function \( f(x) \).

- Suppose \( f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \)
- Plug in \( x = a \) to get \( c_0 = f(a) \)
- \( \frac{d}{dx} \) gives \( f'(x) = \sum_{n=1}^{\infty} c_n(n)(x-a)^{n-1} \)
- Plug in \( x = a \) to get \( c_1 = f'(a) \)
- \( \frac{d}{dx} \) gives \( f''(x) = \sum_{n=2}^{\infty} c_n(n)(n-1)(x-a)^{n-2} \)
- Plug in \( x = a \) to get \( c_2 = f''(a)/2 \)
- Continue to get \( c_n = \frac{f^{(n)}(a)}{n!} \)

3. Here we find the Taylor Series for \( f(x) = \ln x \) centered at \( x = 1 \).
(a) Fill in the chart below

<table>
<thead>
<tr>
<th>n</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(1) )</th>
<th>( c_n = \frac{f^{(n)}(1)}{n!} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \ln x )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{x} )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( -\frac{1}{x^2} )</td>
<td>-1</td>
<td>( -\frac{1}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{2}{x^3} )</td>
<td>2</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>4</td>
<td>( -\frac{3!}{x^4} )</td>
<td>-3!</td>
<td>( -\frac{1}{4} )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{4!}{x^5} )</td>
<td>4!</td>
<td>( \frac{1}{5} )</td>
</tr>
<tr>
<td>6</td>
<td>( -\frac{5!}{x^6} )</td>
<td>-5!</td>
<td>( -\frac{1}{6} )</td>
</tr>
<tr>
<td>n</td>
<td>( \frac{(-1)^{n-1}(n-1)!}{x^n} )</td>
<td>( (-1)^{n-1}(n-1)! )</td>
<td>( \frac{(-1)^{n-1}}{6} )</td>
</tr>
</tbody>
</table>

(b) Write down the power series (fill in the \( c_n \)):

\[
\ln x = c_0 + c_1(x - 1) + c_2(x - 1)^2 + c_3(x - 1)^3 + c_4(x - 1)^4 + c_5(x - 1)^5 + \cdots
\]

\[
= (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \frac{1}{5}(x - 1)^5 - \frac{1}{6}(x - 1)^6 + \cdots
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 1)^n}{n}
\]

4. **Clicker** Given a function, \( f(x) \), it is always possible to find a Taylor Series that equals \( f(x) \).

(a) True  (b) False  **Correct**

**Solution:** But, the answer is actually “mostly true”.

Here are some example that cause problems

\[
f(x) = \begin{cases} 
1 & \text{if } x < 0 \\
0 & \text{if } x \geq 0 
\end{cases}
\]

\[
g(x) = \begin{cases} 
0 & \text{if } x = 0 \\
e^{-1/x^2} & \text{if } x \neq 0 
\end{cases}
\]

Note that \( g(x) \) is continuous and infinitely differentiable but the Taylor Series for \( g(x) \) is just 0 (kinda tricky to show this).

5. **Clicker** Suppose \( f(x) \) is a function and \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) is its Taylor Series.

Sometimes it is easier to make computations with a Taylor Polynomial of \( f \) rather than \( f(x) \).

(a) True  **Correct**  (b) False
Important Series to Know (or at least be familiar with):

\[ \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \]

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \]

\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \]

\[ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \]

\[ \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \]

\[ \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \]

(1 + x)^k = (Its in the text but I don’t think its important enough to fuss with.)

6. Use Taylor Series to compute \( \int_{0}^{1/2} \cos(x^2) \, dx \)

(a) Find a series for \( \cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \cdots \)

(b) Integrate your series

\[ \int \cos(x^2) = x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \frac{x^{17}}{17 \cdot 8!} - \cdots \]

(c) Integrate your series

\[ \int_{0}^{1/2} \cos(x^2) = \frac{1}{2} - \frac{1}{5 \cdot 2^3 \cdot 2!} + \frac{1}{9 \cdot 2^5 \cdot 4!} - \frac{1}{13 \cdot 2^{13} \cdot 6!} + \frac{1}{17 \cdot 2^{17} \cdot 8!} - \cdots \]

(d) Approximate your series by taking a couple terms. How accurate are you?

Solution:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( S_n )</th>
<th>Max Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.003125</td>
</tr>
<tr>
<td>2</td>
<td>0.496875</td>
<td>0.00000904</td>
</tr>
<tr>
<td>3</td>
<td>0.49688404</td>
<td>1.3 \times 10^{-8}</td>
</tr>
<tr>
<td>4</td>
<td>0.49688403</td>
<td>1.1 \times 10^{-11}</td>
</tr>
</tbody>
</table>

7. Find the Taylor Series for \( f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5 \) centered at \( x = 0 \).

Solution:

\[ 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5 \]
Let's graph some of these Taylor Polynomials.

$T_1$: Red  
$T_2$: Blue  
$T_3$: Yellow  
$T_4$: Green  
$T_5 = f$: Black

8. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 1$.

Solution:

$$8 + 16(x - 1) + 26(x - 1)^2 + 17(x - 1)^3 + 2(x - 1)^4 - (x - 1)^5$$
Let's graph some of these Taylor Polynomials.

\( T_1 \): Red
\( T_2 \): Blue
\( T_3 \): Yellow
\( T_4 \): Green
\( T_5 = f \): Black