Solutions

Dec. 2nd: 10.10 : Taylor and Maclaurin Series

In-Class Exercises

Defn. The *Taylor Series for f centered at c* is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

The *Maclaurin Series for f* is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots,$$

namely it is the Taylor Series for f centered at 0.

1. Compute the first 4 terms of the Taylor series for $f(x) = \sqrt{x+1} = (x+1)^{1/2}$ with center 0.

$f'(x) = 1/2(x+1)^{-1/2}$ $\Rightarrow$ $f(0)=1$ $\Rightarrow$ $a_0 = 1$

$f''(x) = -1/4(x+1)^{-3/2}$ $\quad f'(0) = 1/2 \quad a_1 = 1/2$

$f^{(3)}(x) = 3/8(x+1)^{-5/2}$ $\quad f''(0) = -1/4 \quad a_2 = -1/8$

$f^{(3)}(0) = 3/8 \quad a_3 = 3/48$

1st 4 terms: $1 + 1/2x - 1/8x^2 + 3/48x^3$

2. Compute the Taylor series for $f(x) = \frac{1}{x}$ centered at 2.

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f'''(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5}$

$\vdots$

$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$

$\Rightarrow \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n n!\, 2^{-n+1}}{n!} = \frac{(-1)^n}{2^{n+1}}$

$\Rightarrow$ Taylor series centered at 2 $= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$

3. (Clicker) Compute the Maclaurin series for $f(x) = e^x$.

a. $\sum_{n=0}^{\infty} nx^n$

b. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

c. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

d. $\sum_{n=0}^{\infty} \frac{e \cdot x^n}{n!}$

$f^{(n)}(x) = e^x \Rightarrow \frac{f^{(n)}(0)}{n!} = \frac{e^0}{n!} = \frac{1}{n!}$

$\Rightarrow$ Maclaurin series $= \sum_{n=0}^{\infty} \frac{x^n}{n!}$

4. Compute the Maclaurin series for $f(x) = \sin x$.

$f(x) = \sin x \quad f(0) = 0$
$f'(x) = \cos x \quad f'(0) = 1$
$f''(x) = -\sin x \quad f''(0) = 0$
$f^{(3)}(x) = -\cos x \quad f^{(3)}(0) = -1$
$f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$
$f^{(5)}(x) = \cos x \quad f^{(5)}(0) = 1$
$\vdots \qquad \vdots$

$\Rightarrow f^{(n)}(0) = \begin{cases} 0 & n \text{ even} \\ 1 \text{ or } -1 & n \text{ odd} \end{cases}$

n odd $\Rightarrow n = 2k+1$ & $f^{(2k+1)}(0) = (-1)^k$

even #'s

$$\text{M.S.} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

5. (Clicker) Fix any number k. Compute the Maclaurin series for $f(x) = (1+x)^k$.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{k \cdot (k-1) \cdot (k-2) \cdots (k-(n-1)))}{n!} x^n$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{k \cdot (k-1) \cdot (k-2) \cdots (k-n)}{n!} x^n$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{k!}{n!} x^n$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{k}{n!} x^n$

$f'(x) = k(1+x)^{k-1}$
$f''(x) = k(k-1)(1+x)^{k-2}$
$\vdots$
$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-(n-1))(1+x)^{k-n}$

$\Rightarrow \dfrac{f^{(n)}(0)}{n!} = \dfrac{k(k-1)(k-2)\cdots(k-(n-1))}{n!}$

$\therefore$ Maclaurin series $= \displaystyle\sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!}$

Defn. The N^{th} *degree Taylor polynomial of f with center c* is the polynomial:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(N)}(c)}{N!}(x-c)^N.$$

The Remainder $R_N(x)$ of Taylor series is defined as

$$R_N(x) = f(x) - T_N(x).$$

6. (Clicker) Compute the 3^{rd} degree Taylor polynomial $T_3(x)$ of $f(x) = \sqrt{x+1}$ with center $c = 0$.

a. $1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{24}$

b. $1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48}$

c. $1 + \frac{x}{2} - \frac{x^2}{8}$

d. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{3x^3}{48}$

By #1, the 1st 4 terms of the Taylor series is

$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{3x^3}{48}$,

which gives $T_3(x)$.