Section 11.10: Taylor Series

- Goal: Given a function: find a power series that equals the function
- Taylor Series
  \[ f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ where } c_n = \frac{f^{(n)}(a)}{n!}. \]
- Maclaurin Series (Taylor Series with \(a = 0\))
  \[ f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = \frac{f^{(n)}(0)}{n!}. \]

Warm-up Problems

1. **[Clicker]** (Method: multiplication)

   \[ \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \]

   Use multiplication of power series to find a series for \( \frac{1}{2+x} \cdot \ln(1 + x). \)

   (This is definitely a pain, but just distribute the terms out.)

   \[
   \frac{1}{2+x} \cdot \ln(1 + x) = \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \right) \left( \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \cdots \right)
   \]

   \[
   = x - \frac{x^2}{2} + \frac{5x^3}{12} - \frac{x^4}{3} + \frac{4x^5}{15} - \frac{13x^6}{60} + \frac{151x^7}{840} - \frac{16x^8}{105} + \frac{83x^9}{630} - \frac{73x^{10}}{630} + \cdots
   \]

   (a) \( \sum_{n=1}^{\infty} \frac{1}{2n+1} x^n \)

   (b) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} x^n \)

   (c) \( \sum_{n=1}^{\infty} \frac{1}{2n+1} x^n \)

   (d) \( \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n+1} x^n \)

   (e) My clicker is a victim of voter fraud!

   **Solution:** It is VERY difficult to find a pattern when using the multiplication (or division) technique—don’t even try! Just multiply it out, distribute and get as many terms as you need.

Lecture Notes:

Class Problems

2. Let \( f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \)

   (a) Find \( f(0) = 1 \)

   (b) Find \( f(1) = e \)
Solution: You can compute partial sums to get an idea of this answer.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$S_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.716666666666667</td>
</tr>
<tr>
<td>10</td>
<td>2.718281801146384</td>
</tr>
<tr>
<td>11</td>
<td>2.718281826198493</td>
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<tr>
<td>20</td>
<td>2.718281828459045</td>
</tr>
<tr>
<td>100</td>
<td>2.718281828459045</td>
</tr>
<tr>
<td>$e$</td>
<td>2.718281828459045</td>
</tr>
</tbody>
</table>

(c) Find $f'(x) = f(x)$
(d) Find $f''(x) = f(x)$
(e) Graph $f(x)$. 
Lecture Notes: You can’t (easily) graph \( f(x) \) without knowing more, but you can graph partial sums. So we graph the following:

\[
\begin{align*}
f_0 &= 1 \\
f_1 &= 1 + x \quad \text{(Blue)} \\
f_2 &= 1 + x + \frac{x^2}{2!} \quad \text{(Yellow)} \\
f_3 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad \text{(Green)} \\
f_4 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \\
f_5 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \quad \text{(Red)} \\
f_6 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \\
f_7 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \\
f_8 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}
\end{align*}
\]

Lecture Notes: The goal now is:

Given a function \( f(x) \), find a representation of \( f(x) \) with power series.
The idea is similar to:

- Let \( f(x) = \frac{1}{1-x} \)
- Then \( f(x) \) can also be represented as \( f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots \).

(Note: There will always be the question of radius of convergence, interval of convergence but these issues confound the main point above.

Idea of attack for a starting function \( f(x) \).

- Suppose \( f(x) = \sum_{n=0}^{\infty} c_n x^n \)
- Plug in \( x = 0 \) to get \( c_0 = f(0) \)
- \( \frac{d}{dx} \) gives \( f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1} \)
- Plug in \( x = 0 \) to get \( c_1 = f'(0) \)
- \( \frac{d}{dx} \) gives \( f''(x) = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} \)
- Plug in \( x = 0 \) to get \( c_2 = f''(0)/2 \)
- Continue to get \( c_n = \frac{f^{(n)}(0)}{n!} \)

Do this for \( f(x) = e^x \).

I suggest you set up a table to do this:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(0) )</th>
<th>( c_n = f^{(n)}(0)/n! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( e^x )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( e^x )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( e^x )</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>3</td>
<td>( e^x )</td>
<td>1</td>
<td>1/3!</td>
</tr>
<tr>
<td>4</td>
<td>( e^x )</td>
<td>1</td>
<td>1/4!</td>
</tr>
</tbody>
</table>

Do this for \( f(x) = \ln(1 + x) \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(0) )</th>
<th>( c_n = f^{(n)}(0)/n! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \ln(1 + x) )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( 1/(1 + x) )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( -1/(1 + x)^2 )</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>3</td>
<td>( 2/(1 + x)^3 )</td>
<td>2</td>
<td>2/3!</td>
</tr>
<tr>
<td>4</td>
<td>( -6/(1 + x)^4 )</td>
<td>-3</td>
<td>-3!/4! = -1/4</td>
</tr>
<tr>
<td>5</td>
<td>( 4!/(1 + x)^5 )</td>
<td>4</td>
<td>4!/5! = 1/5</td>
</tr>
</tbody>
</table>

This is called Taylor Series. Well, actually Maclaurin Series since we centered at \( x = 0 \) If we center at some point other than 0, it is Taylor Series centered at \( x = a \).

3. Find the Taylor Series for \( f(x) = \sin x \) centered at \( x = 0 \)
   (Make a table for the derivatives!)

**Solution:**

\[
x = \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots
\]
4. Find the Taylor Series for $f(x) = \cos x$ centered at $x = 0$

Solution:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots$$

5. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 0$.

Solution:

$$4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$$
Let's graph some of these Taylor Polynomials.

$T_1$: Red
$T_2$: Blue
$T_3$: Yellow
$T_4$: Green
$T_5 = f$: Black

6. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 1$.

Solution:

$$8 + 16 \cdot (x - 1) + 26 \cdot (x - 1)^2 + 17 \cdot (x - 1)^3 + 2 \cdot (x - 1)^4 - (x - 1)^5$$
Lets graph some of these Taylor Polynomials.

$T_1$: Red
$T_2$: Blue
$T_3$: Yellow
$T_4$: Green
$T_5 = f$: Black