Math 132 - November 30, 2016

Solutions

Section 11.10: Taylor Series

- Goal: Given a function: find a power series that equal the function
- Taylor Series
  $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ where $c_n = \frac{f^{(n)}(a)}{n!}$.
- Maclaurin Series (Taylor Series with $a = 0$)
  $f(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = \frac{f^{(n)}(0)}{n!}$.

Warm-up Problems

1. $\boxed{\textbf{Clicker}}$ (Method: multiplication)

$$\begin{aligned}\log(1+x) =& x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \\ \frac{1}{2+x} =& \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \cdots\end{aligned}$$

Use multiplication of power series to find a series for $\frac{1}{2+x} \cdot \ln(1+x)$.
(This is definitely a pain, but just distribute the terms out.)

$$\begin{aligned}\frac{1}{2+x} \cdot \ln(1+x) =& \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)\left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} - \cdots\right) \\ =& \frac{x}{2} - \frac{x^2}{2} + \frac{5x^3}{12} - \frac{x^4}{3} + \frac{4x^5}{15} - \frac{13x^6}{60} + \frac{151x^7}{840} - \frac{16x^8}{105} + \frac{83x^9}{630} - \frac{73x^{10}}{630} + \cdots\end{aligned}$$

(a) $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x^n$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$

(c) $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x^n$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{2^{n+1}} x^n$

(e) My clicker is a victim of voter fraud!

Solution: It is VERY difficult to find a pattern when using the multiplication (or division) technique–don't even try!. Just multiply it out, distribute and get as many terms as you need.

Lecture Notes:

Class Problems

2. Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

(a) Find $f(0) = 1$

(b) Find $f(1) = e$

Solution: You can compute partial sums to get an idea of this answer.

n	S_n
5	2.716666666666667
10	2.718281801146384
11	2.718281826198493
20	2.718281828459045
100	2.718281828459045
$e =$	2.718281828459045

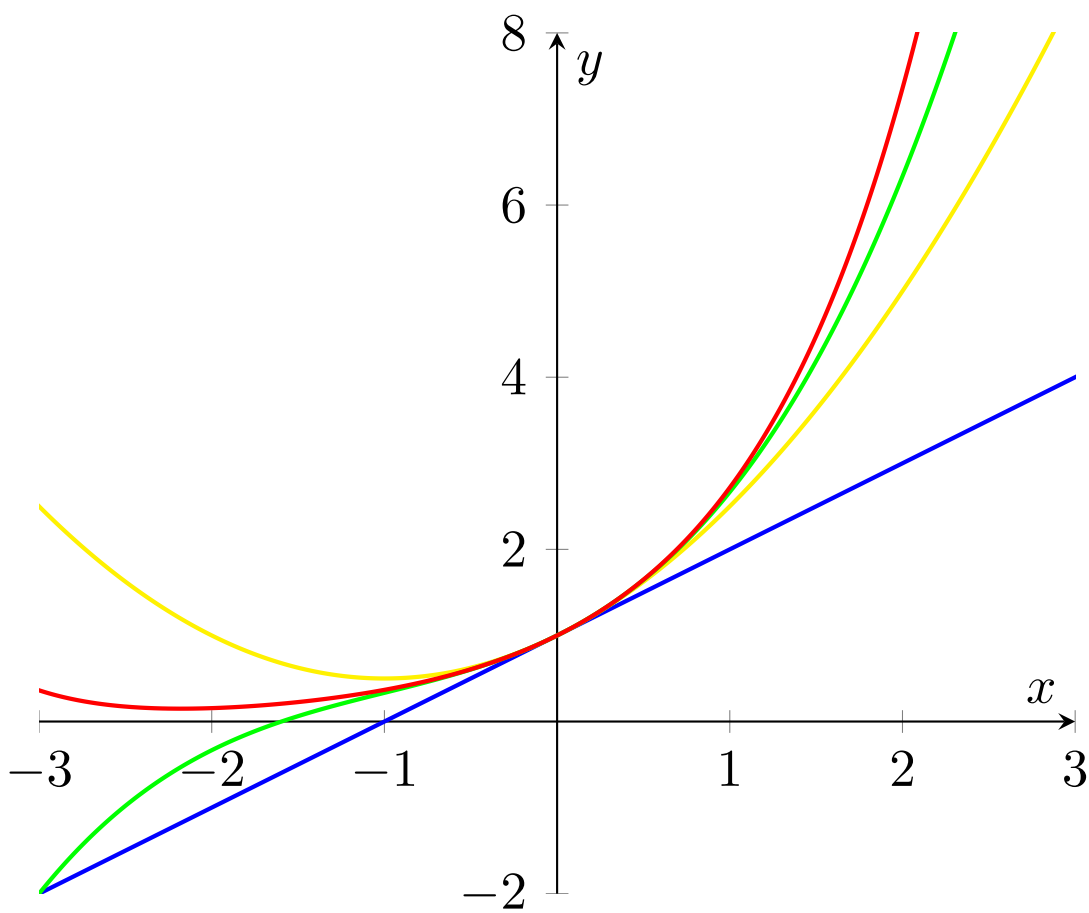
(c) Find $f'(x) = f(x)$

(d) Find $f''(x) = f(x)$

(e) Graph $f(x)$.

Lecture Notes: You can't (easily) graph $f(x)$ without knowing more, but you can graph partial sums. So we graph the following:

$$
\begin{aligned}
f_0 =& 1 \\
f_1 =& 1 + x \qquad \text{(Blue)} \\
f_2 =& 1 + x + \frac{x^2}{2!} \qquad \text{(Yellow)} \\
f_3 =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \qquad \text{(Green)} \\
f_4 =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \\
f_5 =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \\
f_6 =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \qquad \text{(Red)} \\
f_7 =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \\
f_8 =& 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}
\end{aligned}
$$



Lecture Notes: The goal now is:

$\boxed{\text{Given a function } f(x)\text{, find a representation of } f(x) \text{ with power series.}}$

The idea is similar to:

- Let $f(x) = \frac{1}{1-x}$
- Then $f(x)$ can also be represented as $f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$.

(Note: There will always be the question of radius of convergence, interval of convergence but these issue confond the main point above.

Idea of attack for a starting function $f(x)$.

- Suppose $f(x) = \sum_{n=0}^{\infty} c_n x^n$
- Plug in $x = 0$ to get $c_0 = f(0)$
- $\frac{d}{dx}$ gives $f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}$
- Plug in $x = 0$ to get $c_1 = f'(0)$
- $\frac{d}{dx}$ gives $f''(x) = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$
- Plug in $x = 0$ to get $c_2 = f''(0)/2$
- Continue to get $c_n = \frac{f^{(n)}(0)}{n!}$

Do this for $f(x) = e^x$.

I suggest you set up a table to do this:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = f^{(n)}(0)/n!$
0	e^x	1	1
1	e^x	1	1
2	e^x	1	$1/2$
3	e^x	1	$1/3!$
4	e^x	1	$1/4!$

Do this for $f(x) = \ln(1+x)$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$c_n = f^{(n)}(0)/n!$
0	$\ln(1+x)$	0	0
1	$1/(1+x)$	1	1
2	$-1/(1+x)^2$	-1	$1/2$
3	$2/(1+x)^3$	2	$2/3!$
4	$-6/(1+x)^4$	$-3!$	$-3!/4! = -1/4$
5	$4!/(1+x)^5$	$4!$	$4!/5! = 1/5$

This is called Taylor Series. Well, actually Maclaurin Series since we centered at $x = 0$ If we center at some point other than 0, it is Taylor Series centered at $x = a$.

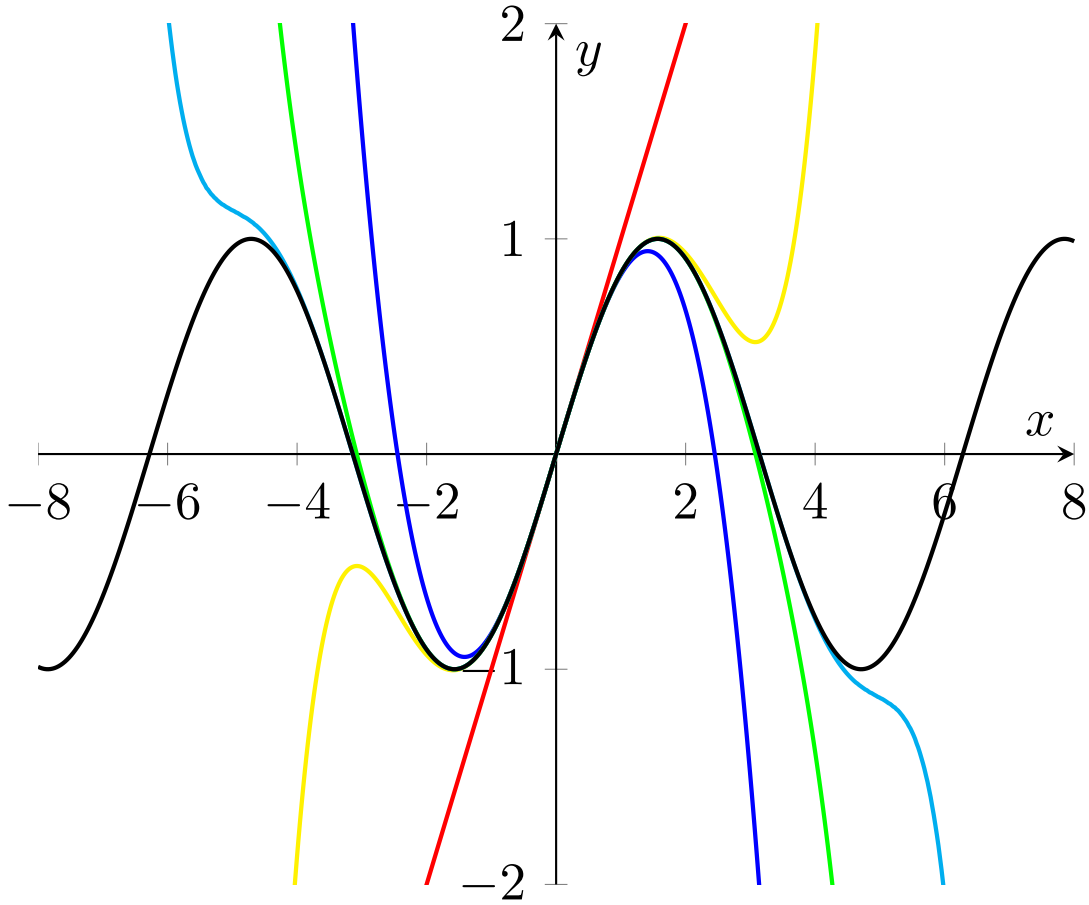
3. Find the Taylor Series for $f(x) = \sin x$ centered at $x = 0$
(Make a table for the derivatives!)

Solution:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} + \cdots$$

Lets graph some of these Taylor Polynomials.

T_1: Red
T_3: Blue
T_5: Yellow
T_7: Green
T_{11}: Cyan
f: Black



4. Find the Taylor Series for $f(x) = \cos x$ centered at $x = 0$

   Solution:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + \cdots$$

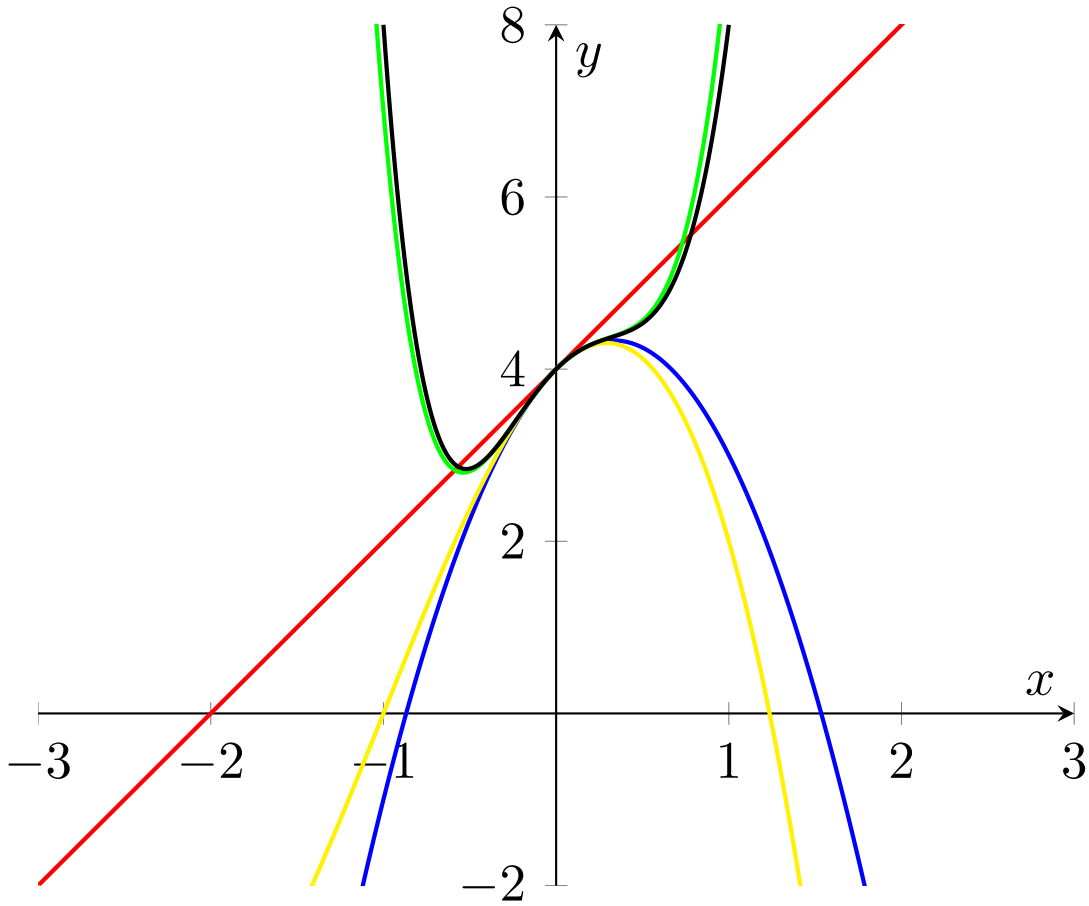
5. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 0$.

   Solution:

$$4 + 2\,x - 3\,x^2 - x^3 + 7\,x^4 - x^5$$

Lets graph some of these Taylor Polynomials.

T_1: Red
T_2: Blue
T_3: Yellow
T_4: Green
$T_5 = f$: Black



6. Find the Taylor Series for $f(x) = 4 + 2x - 3x^2 - x^3 + 7x^4 - x^5$ centered at $x = 1$.

   Solution:

$$8 + 16 * (x-1) + 26 * (x-1)^2 + 17 * (x-1)^3 + 2 * (x-1)^4 - (x-1)^5$$

Lets graph some of these Taylor Polynomials.

T_1: Red
T_2: Blue
T_3: Yellow
T_4: Green
$T_5 = f$: Black