Math 132 - November 28, 2016

Solutions

Section 11.9: Power Series as Functions

- Section 11.8: Radius and Interval of Convergence
- Operations on Power Series: Substitution, Addition, Multiplication, Division (difficult), Differentiation, Integration

Warm-up Problems

1. $\boxed{\textbf{Clicker}}$ Given the sum below:

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \cdots = \frac{1}{1-x}$$

What is the sum:

$$1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 + \cdots$$

(a) $\frac{1}{1-x}$ (b) $\frac{1}{1+x}$ $\boxed{\textbf{Correct}}$ (c) $\frac{1}{1-x^2}$ (d) $\frac{1}{1+x^2}$ (e) Diverges

Solution: If $f(x) = \frac{1}{1-x}$ then note the second series is $f(-x) = \frac{1}{1+x}$.

Lecture Notes: Known Power Series: we really only have one big example where we can find the sum:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

In section 11.10, we will learn how to represent any function as a power series. But for now, this is our main example.
Main idea of the section:

$\boxed{\text{Manipulate a known power series to get new power series.}}$

The methods of manipulation are:

- Addition
- Subtraction
- Multiplication (difficult)
- Division (long division, more difficult)
- Differentiation
- Integration

Some of these examples are easy and silly but they illustrate the techniques. (There are often better/easier ways to calculate these series).
Key point is be familiar with these techniques and be able to use them.

$$\frac{1}{1+x} - \frac{1}{1-x} = \frac{2x}{x^2-1}$$

$$\sum_{n=0}^{\infty}(-1)^n x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}((-1)^n - 1)x^n$$

$$= -2x - 2x^3 - 2x^5 - 2x^7 - 2x^9 = -2\sum_{n=0}^{\infty} x^{2n+1}$$

Here's another way to do the same thing (probably better, definitely more "usual"):

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \qquad \text{(Substitution)}$$
$$\frac{2x}{1-x^2} = 2x\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} 2x^{2n+1}$$
$$\frac{2x}{x^2-1} = -\sum_{n=0}^{\infty} 2x^{2n+1}$$

Differentiation. Note: reindexing the sum is tricky!
Best advice: write out the series term by term and watch for patterns.

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$$
$$\frac{1}{(1-x)^2} = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=0}^{\infty} nx^{n-1}$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

Class Problems

For the problems below, work hard to get the correct representation of the series. To do this, write out the series term-by-term and then try to find the pattern.

2. (Method: Substitution)
   Find a series for $\frac{1}{1+x^2}$

   Solution:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} - x^{12} + \cdots$$

3. $\boxed{\textbf{Clicker}}$ Find a power series for $f(x) = \dfrac{x^3}{4+x}$

   (a) $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n x^n$

   (b) $\displaystyle\sum_{n=0}^{\infty} 4^n x^n$

   (c) $\displaystyle\sum_{n=0}^{\infty}(-4)^{n+1} x^{n+3}$

   (d) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}} x^{n+3}$ $\boxed{\textbf{Correct}}$

   (e) There is no series for $f(x)$.

Solution:

$$\begin{aligned}\frac{x^3}{4+x} =& x^3 \cdot \frac{1/4}{1-(-x/4)} \\ =& \frac{1}{4}x^3 \sum_{n=0}^{\infty} \left(-\frac{x}{4}\right)^n \\ =& \frac{1}{4}x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} x^n \\ =& \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{n+3}\end{aligned}$$

4. Reindex the solution from the previous question so that the power of x is x^n.

   Solution:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{n+3} =& \sum_{n=3}^{\infty} \frac{(-1)^{n-3}}{4^{n-2}} x^n \\ =& 16 \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{4^n} x^n\end{aligned}$$

5. (Method: Differentiation)
   Find a series for $\frac{1}{(1-x)^3}$

   Solution:

$$\begin{aligned}\frac{1}{(1-x)^3} =& \frac{1}{2} \cdot \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) \\ =& \frac{1}{2} \cdot \frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} x^n\right) \\ =& \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{1}{2}\sum_{n=0}^{\infty}(n+2)(n+1)x^n\end{aligned}$$

6. (Method: Integration. Note $\int \frac{1}{1+x^2}\, dx = \arctan x + C$)
   Find a series for $\frac{1}{1+x^2}$

   Solution:

$$\begin{aligned}\arctan x =& \int \frac{1}{1+x^2}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \\ =& x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots\end{aligned}$$

   Note, you can check to make sure C is correct by plugging in $x = 0$ on both sides of the equation. (This will definitely get you sometimes!)

7. Use your previous series to find a series for $\arctan(1) = \pi/4$

   Solution: The series converges for $x = 1$ (why?).

$$\begin{aligned}\arctan 1 = \frac{\pi}{4} =& \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \\ =& 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\end{aligned}$$

Convergence is terribly slow!

n	$4S_n$	$\lvert\pi - 4S_n\rvert$
10	3.232315809405592	0.0907
100	3.15149340107099	0.0099
1000	3.142591654339543	0.000999

8. Find a series for $\frac{1}{1+x^{10}}$ and use your series to approximate $\displaystyle\int_0^{0.25} \frac{1}{1+x^{10}}\,dx$

Solution:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 - x^6 + \cdots$$

$$\frac{1}{1+x^{10}} = \sum_{n=0}^{\infty}(-1)^n x^{10n} = 1 - x^{10} + x^{20} - x^{30} + x^{40} - x^{50} - x^{60} + \cdots$$

$$\int \frac{1}{1+x^{10}}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1} x^{10n+1} = x - \frac{1}{11}x^{11} + \frac{1}{21}x^{21} - \frac{1}{31}x^{31} + \frac{1}{41}x^{41} - \frac{1}{51}x^{50} - \frac{1}{61}x^{60} + \cdots$$

$$\int_0^{1/4} \frac{1}{1+x^{10}}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1}\left(\frac{1}{4}\right)^{10n+1} = \frac{1}{4} - \frac{1}{11}\left(\frac{1}{4}\right)^{11} +$$

Note: no need to find C in the integral since we know we're going to plug in $x = 0.25$ and $x = 0$.

This is an alternating series. So, the integral is equal to $1/4$, accurate to $\frac{1}{11} \cdot \frac{1}{4^{11}} \approx 2 \times 10^{-8}$.

9. (Method: multiplication)
Use multiplication of power series to find a series for $\frac{1}{1-x} \cdot \frac{1}{1-x}$.
(This is definitely a pain, but just distribute the terms out.)

$$(1 + x + x^2 + x^3 + x^4 + \cdots)(1 + x + x^2 + x^3 + x^4 + \cdots) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

10. (Method: Long Division–Challenging and worth avoiding whenever possible)
Use long division of power series to find a series for $\frac{\ln(1+x)}{\ln(1-x)}$
Note: $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$
and: $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \cdots$

Solution: Tricky! This won't be tested but basically just use long division (I couldn't figure out a nice way to typeset the long division so no solution here.)

11. Find power series for the following functions (you figure out the method(s) to use!)

(a) $\displaystyle\frac{1}{4+3x} = \frac{1/4}{1+\frac{3}{4}x} = \sum_{n=0}^{\infty} \frac{1}{4}\left(-\frac{3x}{4}\right)^n$

(b) $\displaystyle\frac{x}{(1+x)^2} = x \cdot D_x\left(-\frac{1}{1+x}\right)$

(c) $\displaystyle\frac{1+2x}{1-x} = \frac{1}{1-x} + 2x\left(\frac{1}{1-x}\right)$