Warm-up Problems

1. **Clicker** Given the sum below:

\[ 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \cdots = \frac{1}{1-x} \]

What is the sum:

\[ 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 + \cdots \]

(a) \( \frac{1}{1-x} \) (b) \( \frac{1}{1+x} \) **Correct** (c) \( \frac{1}{1-x^2} \) (d) \( \frac{1}{1+x^2} \) (e) Diverges

**Solution:** If \( f(x) = \frac{1}{1-x} \) then note the second series is \( f(-x) = \frac{1}{1+x} \).

**Lecture Notes:** Known Power Series: we really only have one big example where we can find the sum:

\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \]

In section 11.10, we will learn how to represent any function as a power series. But for now, this is our main example. Main idea of the section:

**Manipulate a known power series to get new power series.**

The methods of manipulation are:

- Addition
- Subtraction
- Multiplication (difficult)
- Division (long division, more difficult)
- Differentiation
- Integration

Some of these examples are easy and silly but they illustrate the techniques. (There are often better/easier ways to calculate these series).

Key point is be familiar with these techniques and be able to use them.

\[ \frac{1}{1+x} - \frac{1}{1-x} = \frac{2x}{x^2-1} \]

\[ \sum_{n=0}^{\infty} (-1)^n x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} ((-1)^n - 1)x^n \]

\[ = -2x - 2x^3 - 2x^5 - 2x^7 - 2x^9 = -2 \sum_{n=0}^{\infty} x^{2n+1} \]
Here’s another way to do the same thing (probably better, definitely more “usual”):

\[
\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \quad \text{(Substitution)}
\]

\[
\frac{2x}{1-x^2} = 2x \sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} 2x^{2n+1}
\]

\[
\frac{2x}{x^2-1} = -\sum_{n=0}^{\infty} 2x^{2n+1}
\]

Differentiation. Note: reindexing the sum is tricky!
Best advice: write out the series term by term and watch for patterns.

\[
\left( \frac{1}{1-x} \right)' = \frac{1}{(1-x)^2}
\]

\[
\frac{1}{(1-x)^2} = \left( \sum_{n=0}^{\infty} x^n \right)' = \sum_{n=0}^{\infty} n x^{n-1}
\]

\[
= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n
\]

Class Problems
For the problems below, work hard to get the correct representation of the series. To do this, write out the series term-by-term and then try to find the pattern.

2. (Method: Substitution)
Find a series for \(\frac{1}{1+x^2}\)

Solution:
\[
\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} - x^{12} + \cdots
\]

3. Clicker Find a power series for \(f(x) = \frac{x^3}{4+x}\)

(a) \(\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n x^n\)

(b) \(\sum_{n=0}^{\infty} 4^n x^n\)

(c) \(\sum_{n=0}^{\infty} (-4)^{n+1} x^{n+3}\)

(d) \(\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{n+3} \quad \text{Correct}\)

(e) There is no series for \(f(x)\).
Solution:
\[ \frac{x^3}{4 + x} = x^3 \cdot \frac{1/4}{1 - (-x/4)} \]
\[ = \frac{1}{4} x^3 \sum_{n=0}^{\infty} (-x/4)^n \]
\[ = \frac{1}{4} x^3 \sum_{n=0}^{\infty} (-1)^n x^n \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{n+3} \]

4. Reindex the solution from the previous question so that the power of \( x \) is \( x^n \).

Solution:
\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{n+3} = \sum_{n=3}^{\infty} \frac{(-1)^{n-3}}{4^{n-2}} x^n \]
\[ = 16 \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{4^n} x^n \]

5. (Method: Differentiation)
Find a series for \( \frac{1}{(1-x)^3} \)

Solution:
\[ \frac{1}{(1-x)^3} = \frac{1}{2} \cdot \frac{d^2}{dx^2} \left( \frac{1}{1-x} \right) \]
\[ = \frac{1}{2} \cdot \frac{d^2}{dx^2} \left( \sum_{n=0}^{\infty} x^n \right) \]
\[ = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{1}{2} \sum_{n=0}^{\infty} (n+2)(n+1)x^n \]

6. (Method: Integration. Note \( \int \frac{1}{1+x^2} \, dx = \arctan x + C \))
Find a series for \( \frac{1}{1+x^2} \)

Solution:
\[ \arctan x = \int \frac{1}{1+x^2} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \]
\[ = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots \]

Note, you can check to make sure \( C \) is correct by plugging in \( x = 0 \) on both sides of the equation. (This will definitely get you sometimes!)

7. Use your previous series to find a series for \( \arctan(1) = \pi/4 \)

Solution: The series converges for \( x = 1 \) (why?).

\[ \arctan 1 = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \]
\[ = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \]
Convergence is terribly slow!

| $n$  | $4S_n$ | $|\pi - 4S_n|$ |
|------|--------|---------------|
| 10   | 3.232315809405592 | 0.0907       |
| 100  | 3.151493401070999 | 0.0099       |
| 1000 | 3.142591654339543 | 0.000999     |

8. Find a series for $\frac{1}{\sqrt{\pi}}$ and use your series to approximate $\int_0^{0.25} \frac{1}{1 + x^{10}} \, dx$

**Solution:**

\[
\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots
\]

\[
\frac{1}{1 + x^{10}} = \sum_{n=0}^{\infty} (-1)^n x^{10n} = 1 - x^{10} + x^{20} - x^{30} + x^{40} - x^{50} - x^{60} + \cdots
\]

\[
\int \frac{1}{1 + x^{10}} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1} x^{10n+1} = x - \frac{1}{11} x^{11} + \frac{1}{21} x^{21} - \frac{1}{31} x^{31} + \frac{1}{41} x^{41} - \frac{1}{51} x^{51} - \frac{1}{61} x^{61} + \cdots
\]

\[
\int_0^{1/4} \frac{1}{1 + x^{10}} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{10n+1} \left( \frac{1}{4} \right)^{10n+1}
\]

Note: no need to find $C$ in the integral since we know we’re going to plug in $x = 0.25$ and $x = 0$.

This is an alternating series. So, the integral is equal to $1/4$, accurate to $1/11 \cdot \frac{1}{4}^{11} \approx 2 \times 10^{-8}$.

9. (Method: multiplication)

Use multiplication of power series to find a series for $\frac{1}{1-x} \cdot \frac{1}{1+x}$.

(This is definitely a pain, but just distribute the terms out.)

\[
(1 + x + x^2 + x^3 + x^4 + \cdots)(1 + x^2 + x^3 + x^4 + \cdots) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots
\]

10. (Method: Long Division–Challenging and worth avoiding whenever possible)

Use long division of power series to find a series for $\frac{\ln(1+x)}{\ln(1-x)}$

Note: $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$

and: $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \cdots$

**Solution:** Tricky! This won’t be tested but basically just use long division (I couldn’t figure out a nice way to typeset the long division so no solution here.)

11. Find power series for the following functions (you figure out the method(s) to use!)

(a) \[ \frac{1}{4 + 3x} = \frac{1/4}{1 + \frac{3}{4}x} = \sum_{n=0}^{\infty} \frac{1}{4} \left( \frac{3x}{4} \right)^n \]

(b) \[ \frac{x}{(1+x)^2} = x \cdot D_x \left( \frac{1}{1 + x} \right) \]

(c) \[ \frac{1+2x}{1-x} = \frac{1}{1-x} + 2x \left( \frac{1}{1-x} \right) \]