Answer Key

Nov. 7th: 11.4 : The Comparison Tests

Warm-up / Review

1. Determine whether the following series converge or diverge. If possible, determine the value of the series.

a. $\sum_{n=1}^{\infty} \frac{3}{n^{\frac{2}{3}}}$

p series with $p = 2/3 \leq 1$

$\therefore$ the series $\boxed{\text{diverges}}$

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b. $\sum_{n=1}^{\infty} \frac{n^2}{1+2n^2}$

$\lim_{n\to\infty} \frac{n^2}{1+2n^2} = \frac{1}{2} \neq 0$

$\therefore$ the series $\boxed{\text{diverges}}$ by N^{th} term test

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c. $\sum_{n=1}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right)\right)$ Telescoping! $\boxed{\text{Converges \& equals 2}}$

$S_N = \cos(\frac{\pi}{1}) - \cos(\frac{\pi}{2}) + \cos(\frac{\pi}{2}) - \cos(\frac{\pi}{3}) \pm \ldots + \cos(\frac{\pi}{N}) - \cos(\frac{\pi}{N+1})$

$= \cos\pi - \cos(\pi/N+1)$

$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \cos\pi - \cos(\pi/N+1) = \cos\pi - \cos(0) = -2.$

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d. $\sum_{n=1}^{\infty} \frac{4^{2n}}{3^{3n+2}} = \sum_{n=1}^{\infty} \frac{16^n}{27^n \cdot 9} = \sum_{n=1}^{\infty} \frac{1}{9}\left[\frac{16}{27}\right]^n$ $\quad \left|\frac{16}{27}\right| < 1 \therefore$ converges

$= \frac{1/9 \cdot \left[\frac{16}{27}\right]}{1 - 16/27} = \frac{\frac{16}{9}}{11} = \frac{16}{99}$

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e. $\sum_{n=2}^{\infty} \frac{2n}{n^2+1}$ $\quad f(x) = \frac{2x}{x^2+1}$ $\quad f'(x) = \frac{2(x^2+1)-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2} < 0$ on $[2,\infty)$

$\Rightarrow f$ positive, continuous, decreasing on $[2,\infty)$

$\int_2^{\infty} \frac{2x}{x^2+1}\,dx = \lim_{t\to\infty} \int_2^t \frac{2x}{x^2+1}\,dx = \lim_{t\to\infty} \ln|x^2+1|\Big|_2^t = \lim_{t\to\infty} \ln|t^2+1| - \ln 5 = \infty$

Since the integral diverges, the integral Test implies that the series also diverges.

In-Class Exercises

1. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges.

$\frac{1}{n^2+1} < \frac{1}{n^2}$ Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges.

The Comparison Test. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. Then

i. If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. If $\sum_{n=1}^{\infty} b_n$ diverges and $b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

WORK on next pages

2. (Clicker) Which of the following series converge?

$$I. \sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2} \qquad II. \sum_{n=3}^{\infty} \frac{1}{n-2}$$

a. Neither of them.

b. I only

c. II only

d. I and II

3. (Clicker) Which of the following series converge?

$$I. \sum_{n=1}^{\infty} \frac{5^{n+2}}{3^n - 1} \qquad II. \sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$$

a. Neither of them.

b. I only

c. II only

d. I and II

4. (Clicker) Which of the following series converge?

$$I. \sum_{n=3}^{\infty} \frac{\arctan(n)}{\sqrt{n}-1} \qquad II. \sum_{n=1}^{\infty} \frac{e^n + n}{2^{2n} + n^2}$$

a. Neither of them.

b. I only

c. II only

d. I and II

#2 I.

$$\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2}$$

B/c $\frac{\cos^2(n)}{n^2} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges b/c it's a p-series w/ $p>1$, the Comparison Test implies that $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2}$ also $\boxed{\text{converges.}}$

#2 II: $\sum_{n=3}^{\infty} \frac{1}{n-2}$

Note that $\frac{1}{n-2} \geq \frac{1}{n}$. Also $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges b/c it's a p-series with $p=1$. Then the Comparison Test implies $\sum_{n=3}^{\infty} \frac{1}{n-2}$ also $\boxed{\text{diverges.}}$

#3 I: $\sum_{n=1}^{\infty} \frac{5^{n+2}}{3^n-1}$

Note that $\frac{5^{n+2}}{3^n-1} \geq \frac{5^{n+2}}{3^n}$ & $\sum_{n=1}^{\infty} \frac{5^{n+2}}{3^n} = \sum_{n=1}^{\infty} 25 \cdot \left[\frac{5}{3}\right]^n$ diverges because it's a geometric series w/ $|r| \geq 1$. Then the Comparison Test implies that $\sum_{n=1}^{\infty} \frac{5^{n+2}}{3^n-1}$ also $\boxed{\text{diverges}}$

#3 II: $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$

Note that $\frac{n+1}{n^3+n} \leq \frac{n+n}{n^3} = \frac{2}{n^2}$, & $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges b/c it's a p-series with $p=2>1$. Then the Comparison Test implies that $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$ also $\boxed{\text{converges}}$.

#4 I : $\sum_{n=3}^{\infty} \frac{\arctan(n)}{\sqrt{n}-1}$

Note that $\arctan(x)$ is increasing so if $n \geq 3$,

$\arctan(n) \geq \arctan(3)$.

Then $\frac{\arctan(n)}{\sqrt{n}-1} \geq \frac{\arctan(3)}{\sqrt{n}}$ and $\sum_{n=3}^{\infty} \frac{\arctan(3)}{\sqrt{n}}$ diverges

b/c (since $\arctan(3)$ is a number) it's a p-series with $p=\frac{1}{2}$.

Then the Comparison Test implies that $\sum_{n=1}^{\infty} \frac{\arctan(n)}{\sqrt{n}-1}$ also $\boxed{\text{diverges.}}$

#4 II : $\sum_{n=1}^{\infty} \frac{e^n+n}{2^{2n}+n^2}$

Note that $\frac{e^n+n}{2^{2n}+n^2} \leq \frac{e^n+e^n}{4^n} = 2\cdot\left[\frac{e}{4}\right]^n$.

Then $\sum_{n=1}^{\infty} 2\cdot\left[\frac{e}{4}\right]^n$ converges b/c it's a geometric series with $r=\frac{e}{4}$ & $|r|<1$. By the Comparison Test $\sum_{n=1}^{\infty} \frac{e^n+n}{2^{2n}+n^2}$ converges as well.