Solutions

Nov. 4th: 11.3 : The Integral Test

Warm-Up

1. Compute the following integrals:

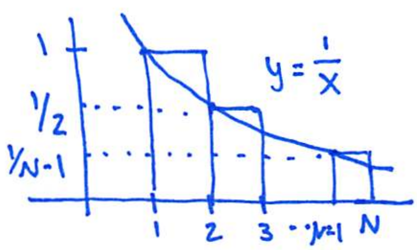
a. $\displaystyle\int_1^N \frac{dx}{x} = \ln x \Big|_1^N = \ln N$

b. $\displaystyle\int_1^\infty \frac{dx}{x^p},\ p \neq 1 = \lim_{t\to\infty} \int_1^t x^{-p}\,dx = \lim_{t\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^t = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}$

$$= \begin{cases} \infty & p<1 \\ \frac{1}{p-1} & p>1 \end{cases}$$

In-Class Exercises

1. Determine whether the harmonic series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ converges or diverges.



$$\int_1^N \frac{1}{x}\,dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N-1}$$
$$\leq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N-1} + \frac{1}{N} = S_N$$

By 1a) $S_N \geq \displaystyle\int_1^N \frac{1}{x}\,dx = \ln N$

Then $\displaystyle\lim_{N\to\infty} S_N \geq \lim_{N\to\infty} \ln N = \infty$ so $\displaystyle\lim_{N\to\infty} S_N = \infty$.

Therefore, the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges.

The Integral Test. Let $f(x)$ be a continuous, positive, decreasing function defined on $[k, \infty)$ and suppose

$$a_n = f(n) \quad \text{for all } n.$$

Then the series $\displaystyle\sum_{n=k}^\infty a_n$ and the integral $\displaystyle\int_k^\infty f(x)\,dx$ have the same behavior. Specifically:

- If $\displaystyle\int_k^\infty f(x)\,dx$ converges, then $\displaystyle\sum_{n=k}^\infty a_n$ converges;
- If $\displaystyle\int_k^\infty f(x)\,dx$ diverges, then $\displaystyle\sum_{n=k}^\infty a_n$ diverges.

2. (Clicker) For which values of $p > 0$ does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

a. $p > 1$

b. $\frac{1}{2} < p < 1$

c. $p \geq 1$

d. $0 < p < 1$

e. $p > 0$

By #1 div. For $p = 1$.

By Integral Test, since $\frac{1}{x^p}$ is continuous, positive, & decreasing on $[1, \infty)$, the series has the same behavior as $\int_1^{\infty} \frac{1}{x^p} dx$; it converges for $p > 1$ & diverges for $0 < p \leq 1$.

3. Use the Integral Test to determine whether the following series converge or diverge.

a. $\sum_{n=0}^{\infty} \frac{1}{9+n^2}$

b. $\sum_{n=1}^{\infty} e^{-n}$

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c. $\sum_{n=1}^{\infty} ne^{-n}$

d. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

d. ~~$\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$~~

#3a) $\sum_{n=0}^{\infty} \frac{1}{9+n^2}$

$f(x) = \frac{1}{9+x^2}$ is continuous, positive, & decreasing on $[0,\infty)$

(to see decreasing, note, $f'(x) = \frac{-2x}{(9+x^2)^2} \leq 0$ For $x \in [0,\infty)$)

By the Integral Test, $\sum_{n=0}^{\infty} \frac{1}{9+n^2}$ & $\int_0^\infty \frac{dx}{9+x^2}$ have the same behavior.

$$\int_0^\infty \frac{dx}{9+x^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{9+x^2} = \lim_{t\to\infty} \int_0^t \frac{1}{9} \frac{dx}{1+(x/3)^2} = \lim_{t\to\infty} \frac{3}{9} \arctan\left(\frac{x}{3}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \frac{1}{3}\arctan(t/3) - 1/3 \arctan 0$$

$$= 1/3 \cdot \pi/2 = \frac{\pi}{6}.$$

B/c $\int_0^\infty \frac{dx}{9+x^2}$ converges, $\sum_{n=0}^{\infty} \frac{1}{9+n^2}$ converges too

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#3b $\sum_{n=1}^{\infty} e^{-n}$

$f(x) = e^{-x}$ is continuous, positive, & decreasing on $[0,\infty)$

To see decreasing, not $f'(x) = -e^{-x} < 0$ For x in $[0,\infty)$.

By the Integral Test, $\sum_{n=1}^{\infty} e^{-n}$ & $\int_1^\infty e^{-x}dx$ have the same behavior.

Since,

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}dx = \lim_{t\to\infty} -e^{-x}\Big|_1^t = \lim_{t\to\infty} -e^{-t} + e^{-1} = e^{-1},$$

the series $\sum_{n=1}^{\infty} e^{-n}$ also converges.

c) $\sum_{n=1}^{\infty} ne^{-n}$

$f(x) = xe^{-x}$ is positive, continuous, & decreasing on $[1,\infty)$ since $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} \leq 0$ for x in $[1,\infty)$. Then

$\sum_{n=1}^{\infty} ne^{-n}$ & $\int_1^{\infty} xe^{-x}dx$ have the same behavior

by the Integral Test.

<u>Note</u>: $\int_1^{\infty} xe^{-x}dx = \lim_{t\to\infty}\int_1^t xe^{-x}dx = \lim_{t\to\infty} -xe^{-x}\Big|_1^t + \int_1^t e^{-x}dx$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$= \lim_{t\to\infty} -te^{-t} + e^{-1} - e^{-x}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{-t}{e^t}^{\infty/\infty} + e^{-1} \underbrace{- e^{-t}}_{\to 0} + e^{-1}$$

Use L'Hopitals Rule

$$= \lim_{t\to\infty} \frac{-1}{e^t} + 2e^{-1} = 2e^{-1}.$$

Since the integral converges, $\sum_{n=1}^{\infty} ne^{-n}$ converges as well.

#3d $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ ③

$f(x) = \frac{1}{x\ln x}$ is positive, continuous, & decreasing on $[2,\infty)$ since

$f'(x) = \frac{-[\ln x + 1]}{[x\ln x]^2} < 0$ for x in $[2,\infty)$.

By the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ & $\int_2^{\infty} \frac{dx}{x\ln x}$ either both converge or both diverge. Note that:

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{t\to\infty} \int_2^{t} \frac{dx}{x\ln x} = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du = \lim_{t\to\infty} \ln|u| \Big|_{\ln 2}^{\ln t}$$

$u = \ln x \quad du = \frac{1}{x}dx$

$$= \lim_{t\to\infty} \ln[\ln t] - \ln[\ln 2] = \infty.$$

Since the integral diverges, $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ also diverges