Terms of the Day

- Increasing sequence
- Decreasing Sequence
- Monotonic Sequence
- Bounded Above Sequence
- Bounded Below Sequence
- Bounded Sequence

Warm-up Problems

1. [Clicker] Find \( \lim_{n \to \infty} \frac{5!(n + 2)!}{(n + 3)!} \)
   
   (a) 0 [Correct] (b) \( \frac{2}{3} \) (c) \( \infty \) (d) \( -\infty \)

2. (From last time) Let \( a_n = x^n \). Find \( \lim_{n \to \infty} a_n \).
   
   **Solution:** This depends on \( x \).
   
   If \( |x| < 1 \) then \( \lim_{n \to \infty} x^n = 0 \)
   
   If \( x = 1 \) then \( \lim_{n \to \infty} x^n = 1 \)
   
   If \( x = -1 \) then \( \lim_{n \to \infty} x^n \) diverges
   
   If \( |x| > 1 \) then \( \lim_{n \to \infty} x^n \) diverges

3. Find the derivative \( \frac{d}{dn} \left( \frac{n + 1}{n - 1} \right) \)

4. (e – ln trick practice) Find \( \lim_{n \to \infty} \left( \frac{n + 1}{n - 1} \right)^n \)

   **Solution:**
   
   \[
   \lim_{n \to \infty} \left( \frac{n + 1}{n - 1} \right)^n = \lim_{n \to \infty} \exp \left[ \ln \left( \frac{n + 1}{n - 1} \right)^n \right] 
   \]
   
   \[
   = \lim_{n \to \infty} \exp \left[ n \cdot \ln \left( \frac{n + 1}{n - 1} \right) \right] 
   \]
   
   \[
   = \lim_{n \to \infty} \exp \left[ \frac{\ln \left( \frac{n + 1}{n - 1} \right)}{1/n} \right] 
   \]
   
   (L’Hopital’s Rule: \( \frac{0}{0} \))
   
   \[
   = \lim_{n \to \infty} \exp \left[ -\frac{2}{(n - 1)^2} \right] 
   \]
   
   \[
   = \lim_{n \to \infty} \exp \left[ \frac{2n^2}{(n - 1)^2} \right] 
   \]
   
   \[
   = \lim_{n \to \infty} \exp \left[ \frac{2n^2}{(n - 1)^2} \right] = e^2 
   \]

   Similarly, from yesterday’s sheet:

   \[
   \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x 
   \]

5. Fun problem 1 (Continued Fractions). Find the limit of the sequence:

\[
a_1 = 1, \ a_2 = 1 + \frac{1}{1}, \ a_3 = 1 + \frac{1}{1 + \frac{1}{1}}, \ a_4 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}, \ a_5 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}, \ etc. 
\]
Solution: Note that if \( L = \lim a_n \) then \( L = 1 + \frac{1}{L} \) and so \( L^2 - L - 1 = 0 \) which gives \( L = \frac{1 + \sqrt{5}}{2} \) so \( L \) has to be equal to \( \frac{1 + \sqrt{5}}{2} \).

6. Fun problem 2 (Continued Fractions). Find the limit of the sequence:

\[
\begin{align*}
a_1 &= 2, \quad a_2 = 2 + \frac{1}{1}, \quad a_3 = 2 + \frac{1}{1 + \frac{1}{2}}, \quad a_4 = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1}}}, \quad a_5 = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1}}}}, \quad a_6 = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1}}}}, \\
a_7 &= 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1}}}}}, \quad a_8 = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1}}}}}}, \quad \text{etc.}
\end{align*}
\]

In this sequence, the “diagonal” of the fractions follows the pattern: \( [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, \ldots] \)

Solution: Amazingly, \( \lim a_n = e \). See:

https://arxiv.org/pdf/math/0601660

or

www.math.mun.ca/~sergey/Research/Misc/contfrac_e.pdf

Class Problems

7. For each of the sequences, find \( \lim_{n \to \infty} a_n \)

(a) Let \( x \) be fixed and let \( a_n = \frac{x^n}{n!} \).

Solution: Technically, the squeeze theorem is needed since

\[
-\frac{|x|^n}{n!} = -\frac{|x|^n}{n!} \leq \frac{x^n}{n!} \leq \frac{|x|^n}{n!}
\]

For this, imagine a huge fixed \( x \), so the numerator is exponential. But, there is some \( M > |x| \) so that eventually (for \( n > M \)):

\[
\frac{|x|^n}{n!} = \frac{|x|^n}{1 \cdot 2 \cdot 3 \cdots M(M+1)(M+2)(M+3) \cdots n} = \frac{|x|^n}{M! \cdot (M+1)(M+2)(M+3) \cdots n} \leq \frac{|x|^n}{M!M^{n-M}} = \frac{|x|^n M^M}{M!M^n} = \left( \frac{M^n}{M!} \right) \left( \frac{|x|}{M} \right)^n
\]

Note that \( \frac{M^n}{M!} \) is just a constant and that since \( |x| < M \), \( \frac{|x|}{M} < 1 \) is just a constant too. So, by Number 2:

\[
\lim_{n \to \infty} \frac{|x|^n}{n!} \leq \lim_{n \to \infty} \left( \frac{M^n}{M!} \right) \left( \frac{|x|}{M} \right)^n = \left( \frac{M^n}{M!} \right) \lim_{n \to \infty} \left( \frac{|x|}{M} \right)^n = 0
\]

(b) \( a_n = \frac{n!}{n^n} \)

Solution: Note that

\[
a_n = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdot n \cdots n} = \left( \frac{1}{n} \right) \left( \frac{2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdots n} \right) \leq \left( \frac{1}{n} \right) \left( \frac{n \cdot n \cdot n \cdots n}{n \cdot n \cdot n \cdots n} \right) = \frac{1}{n}
\]

Thus \( 0 \leq a_n \leq \lim \frac{1}{n} = 0. \)
8. For each of the following sequences, determine if it is bounded (above/below/both), monotonic/increasing/decreasing.

(a) \( a_n = \frac{1}{n} \)
   Solution: Bounded and monotonic decreasing
(b) \( a_n = -\frac{1}{n} \)
   Solution: Bounded and monotonic increasing
(c) \( a_n = (-1)^n \)
   Solution: Bounded but not monotonic
(d) \( a_n = \frac{n + 1}{n} \)
   Solution: Bounded and monotonic decreasing
(e) \( a_n = \frac{n + 2}{n} \)
   Solution: Bounded and monotonic increasing
(f) \( a_n = n^2 \)
   Solution: Monotonic increasing but unbounded
(g) \( a_n = -n^2 \)
   Solution: Monotonic decreasing but unbounded
(h) \( a_n = (-1)^n n^2 \)
   Solution: Unbounded but not monotonic

Monotonic Sequence Theorem: Every bounded monotonic sequence converges.

9. Determine if the following sequence is monotonic and bounded: \( a_n = \frac{5n - 2}{7n + 3} \)
   Solution: Increasing since \( f'(n) = 29/(7n + 3)^2 > 0 \). Bounded since \( 0 < a_n < 5/7 \).

10. Experiment with the following recursive sequences. In particular, write down a few terms, determine (if possible): monotonicity (increasing/decreasing), bounded (above/below), convergence (converge/diverge) and find an explicit formula for the terms.

(a) \( a_1 = 4, a_n = a_{n-1} + 5 \)
   Solution: This is an arithmetic sequence.
   \[ 4, 9, 14, 19, \ldots \]
   \( a_n = -1 + 5n \)
   Increasing, bounded below, unbounded above, diverges
(b) \( a_1 = 16, a_n = \frac{1}{2} a_{n-1} \)
   Solution: This is a geometric sequence.
   \[ 16, 8, 4, 2, 1, \frac{1}{2}, \ldots \]
   \( a_n = \frac{32}{2^n} \)
   Decreasing, bounded, converges to 0.
(c) (From the text) \( a_1 = 1, a_n = \frac{1}{2} (a_{n-1} + 6) \)
   Solution: 1, 2.5, 4.75, 5.375, 5.6875, 5.84375, 5.921875, \ldots
   If \( L = \lim_{n \to \infty} a_n \) then \( L = \frac{1}{2}(L + 6) \). Solving gives \( L = 6 \).
   This doesn’t show that the limit exists. Rather, this shows that if there is a limit, then it has to be equal to 6. To see the limit exists, we use mathematical induction to show that the sequence is bounded and increasing.
   Bounded, \( a_1 < 6 \). If \( a_n < 6 \) then \( a_{n+1} = \frac{1}{2}(a_n + 6) < \frac{1}{2}(6 + 6) = 6 \)
   Increasing, true that \( a_2 > a_1 \). Suppose we know that \( a_{k+1} > a_k \). Then, adding six and dividing by 2 gives \( \frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6) \), which is \( a_{k+2} > a_{k+1} \).
   \( a_1 = 0, a_2 = 1, a_{n+1} = a_n + 2a_{n-1} \)
   Solution: 0, 1, 1, 3, 5, 11, 21, 43, \ldots
   Increasing, unbounded, diverges,
   (e) For the sequence from Problem 10d find \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \)
   Solution: Note that this sequence is undefined, 1, 3, \frac{5}{2}, \frac{11}{5}, \ldots

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And, \( \frac{a_{n+1}}{a_n} = 1 + \frac{2}{a_n/a_{n-1}} \) Thus, if \( L = \lim a_{n+1}/a_n \) then \( L = 1 + 2/L \) and \( L^2 = L + 2 \).

\( L^2 - L - 2 = (L - 2)(L + 1) \) and we must have \( L = 2 \).

(Can you find a way to make \( L = -1 \)?)

As with previous question, this only shows that if the limit exists, it must be equal to 2. To see that the limit exists, more work is needed (much more difficult!).