Math 132 - October 28, 2016

Sections: 11.1

Solutions

Terms of the Day

- Increasing sequence
- Decreasing Sequence
- Monotonic Sequence
- Bounded Above Sequence
- Bounded Below Sequence
- Bounded Sequence

Warm-up Problems

1. $\boxed{\textbf{Clicker}}$ Find $\lim_{n\to\infty} \frac{5!(n+2)!}{(n+3)!}$

   (a) 0 $\boxed{\textbf{Correct}}$ (b) $\frac{2}{3}$ (c) ∞ (d) $-\infty$

2. (From last time) Let $a_n = x^n$. Find $\lim_{n\to\infty} a_n$.
   Solution: This depends on x.
   If $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$
   If $x = 1$ then $\lim_{n\to\infty} x^n = 1$
   If $x = -1$ then $\lim_{n\to\infty} x^n$ diverges
   If $|x| > 1$ then $\lim_{n\to\infty} x^n$ diverges

3. Find the derivative $\frac{d}{dn}\left(\frac{n+1}{n-1}\right) = -\frac{2}{(n-1)^2}$

4. ($e - \ln$ trick practice) Find $\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n$

   Solution:

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{n+1}{n-1}\right)^n &= \lim_{n\to\infty} \exp\left[\ln\left(\frac{n+1}{n-1}\right)^n\right] \\
&= \lim_{n\to\infty} \exp\left[n \cdot \ln\left(\frac{n+1}{n-1}\right)\right] \\
&= \lim_{n\to\infty} \exp\left[\frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n}\right] \qquad \text{(L'Hopital's Rule: } \frac{0}{0}\text{)} \\
&= \lim_{n\to\infty} \exp\left[\frac{-2/(n-1)^2}{-1/n^2}\right] \\
&= \lim_{n\to\infty} \exp\left[\frac{2n^2}{(n-1)^2}\right] \\
&= \lim_{n\to\infty} \exp\left[\frac{2n^2}{(n-1)^2}\right] = e^2
\end{aligned}$$

   Similarly, from yesterday's sheet:

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

5. Fun problem 1 (Continued Fractions). Find the limit of the sequence:
   $a_1 = 1$, $a_2 = 1 + \frac{1}{1}$, $a_3 = 1 + \cfrac{1}{1 + \cfrac{1}{1}}$, $a_4 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$, $a_5 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$, etc.

Solution: Note that if $L = \lim a_n$ then $L = 1 + \frac{1}{L}$ and so $L^2 - L - 1 = 0$ which gives $L = \frac{1 \pm \sqrt{5}}{2}$ so L has to be equal to $\frac{1+\sqrt{5}}{2}$.

6. Fun problem 2 (Continued Fractions). Find the limit of the sequence:

$$a_1 = 2,\ a_2 = 2 + \frac{1}{1},\ a_3 = 2 + \cfrac{1}{1 + \cfrac{1}{2}},\ a_4 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}},\ a_5 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1}}}},\ a_6 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}},$$

$$a_7 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1}}}}}},\ a_8 = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}},\ \text{etc.}$$

In this sequence, the "diagonal" of the fractions follows the pattern: $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, \ldots]$

Solution: Amazingly, $\lim a_n = e$. See:
`https://arxiv.org/pdf/math/0601660`
or
`www.math.mun.ca/~sergey/Research/Misc/contfrac_e.pdf`

Class Problems

7. For each of the sequences, find $\lim_{n\to\infty} a_n$

   (a) Let x be fixed and let $a_n = \dfrac{x^n}{n!}$.
   Solution: Technically, the squeeze theorem is needed since

$$-\frac{|x|^n}{n!} = -\frac{|x^n|}{n!} \le \frac{x^n}{n!} \le \frac{|x^n|}{n!} = \frac{|x|^n}{n!}$$

   For this, imagine a huge fixed x, so the numerator is exponential. But, there is some $M > |x|$ so that eventually (for $n > M$):

$$\begin{aligned}\frac{|x|^n}{n!} &= \frac{|x|^n}{1 \cdot 2 \cdot 3 \cdots M(M+1)(M+2)(M+3)\cdots n} \\ &= \frac{|x|^n}{M! \cdot (M+1)(M+2)(M+3)\cdots n} \\ &\le \frac{|x|^n}{M! M^{n-M}} = \frac{|x|^n M^M}{M! M^n} = \left(\frac{M^M}{M!}\right)\left(\frac{|x|}{M}\right)^n\end{aligned}$$

   Note that $\frac{M^M}{M!}$ is just a constant and that since $|x| < M$, $\frac{|x|}{M} < 1$ is just a constant too. So, by Number 2,

$$\begin{aligned}\lim_{n\to\infty} \frac{|x|^n}{n!} &\le \lim_{n\to\infty} \left(\frac{M^M}{M!}\right)\left(\frac{|x|}{M}\right)^n \\ &= \left(\frac{M^M}{M!}\right) \lim_{n\to\infty} \left(\frac{|x|}{M}\right)^n = 0\end{aligned}$$

   (b) $a_n = \dfrac{n!}{n^n}$
   Solution: Note that

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots n}{n \cdot n \cdot n \cdot n \cdot n \cdots n} = \left(\frac{1}{n}\right)\left(\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdots n}{n \cdot n \cdot n \cdot n \cdots n}\right) \le \left(\frac{1}{n}\right)\left(\frac{n \cdot n \cdot n \cdot n \cdots n}{n \cdot n \cdot n \cdot n \cdots n}\right) = \frac{1}{n}$$

   Thus $0 \le \lim a_n \le \lim \frac{1}{n} = 0$.

8. For each of the following sequences, determine if it is bounded (above/below/both), monotonic/increasing/decreasing.

   (a) $a_n = \dfrac{1}{n}$
   Solution: Bounded and monotonic decreasing

   (b) $a_n = \dfrac{-1}{n}$
   Solution: Bounded and monotonic increasing

   (c) $a_n = \dfrac{(-1)^n}{n}$
   Solution: Bounded but not monotonic

   (d) $a_n = \dfrac{n+1}{n}$
   Solution: Bounded and monotonic decreasing

   (e) $a_n = \dfrac{n+1}{n+2}$
   Solution: Bounded and monotonic increasing

   (f) $a_n = n^2$
   Solution: Monotonic increasing but unbounded

   (g) $a_n = -n^2$
   Solution: Monotonic decreasing but unbounded

   (h) $a_n = (-1)^n n^2$
   Solution: Unbounded but not monotonic

Monotonic Sequence Theorem: Every bounded monotonic sequence converges.

9. Determine if the following sequence is monotonic and bounded: $a_n = \frac{5n-2}{7n+3}$.
   Solution: Increasing since $f'(n) = 29/(7n+3)^2 > 0$. Bounded since $0 < a_n < 5/7$.

10. Experiment with the following recursive sequences. In particular, write down a few tersm, determine (if possible): monotonicity (increasing/decreasing), bounded (above/below), convergence (converge/diverge) and find an explicit formula for the terms.

    (a) $a_1 = 4$, $a_n = a_{n-1} + 5$
    Solution: This is an *arithmetic* sequence.
    $4, 9, 14, 19, \ldots$.
    $a_n = -1 + 5n$
    Increasing, bounded below, unbounded above, diverges

    (b) $a_1 = 16$, $a_n = \frac{1}{2}a_{n-1}$
    Solution: This is a *geometric* sequence.
    $16, 8, 4, 2, 1, \frac{1}{2}, \ldots$
    $a_n = \frac{32}{2^n}$
    Decreasing, bounded, converges to 0.

    (c) (From the text) $a_1 = 1$, $a_n = \frac{1}{2}(a_{n-1} + 6)$
    Solution: $1, 2.5, 4.75, 5.375, 5.6875, 5.84375, 5.921875, \ldots$.
    If $L = \lim a_n$ then $L = \frac{1}{2}(L + 6)$. Solving gives $L = 6$.
    This doesn't show that the limit exists. Rather, this shows that if there is a limit, then it has to be equal to 6. To see the limit exists, we use mathematical induction to show that the sequence is bounded and increasing.
    Bounded, $a_1 < 6$. If $a_n < 6$ then $a_{n+1} = \frac{1}{2}(a_n + 6) < \frac{1}{2}(6 + 6) = 6$
    Increasing, true that $a_2 > a_1$. Suppose we know that $a_{k+1} > a_k$. Then, adding six and dividing by 2 gives $\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$, which is $a_{k+2} > a_{k+1}$.

    (d) $a_1 = 0$, $a_2 = 1$, $a_{n+1} = a_n + 2a_{n-1}$
    Solution: $0, 1, 1, 3, 5, 11, 21, 43, \ldots$
    Increasing, unbounded, diverges,

    (e) For the sequence from Probelm 10d, find $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$
    Solution: Note that this sequence is undef, $1, 3, \frac{5}{2}, \frac{11}{5}, \ldots$.

And, $\dfrac{a_{n+1}}{a_n} = \dfrac{a_n + 2a_{n-1}}{a_n} = 1 + \dfrac{2}{a_n/a_{n-1}}$ Thus, if $L = \lim a_{n+1}/a_n$ then $L = 1 + 2/L$ and $L^2 = L + 2$. $L^2 - L - 2 = (L-2)(L+1)$ and we must have $L = 2$.

(Can you find a way to make $L = -1$?)

As with previous question, this only shows that if the limit exists, it must be equal to 2. To see that the limit exists, more work is needed (much more difficult!).