Solutions

# Oct. 28th: 11.1 : Sequences

## Warm-Up

1 Determine the limit of each of the following sequences or show that it diverges.

a. $a_n = \frac{n+1}{n}$ $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + 1/n = 1$

b. $b_n = \cancel{\sin}\ \cos(\pi n)$ $-1, 1, -1, 1$ diverges

c. $c_n = \frac{n!}{(n+1)!}$ $\lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{n!(n+1)} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

## In-Class Exercises

1. (Clicker) For which values of $r$ does the sequence $\{a_n\}$ defined by $a_n = r^n$ converge?

a. $r > 1$

b. $0 \le r \le 1$

c. $-1 \le r \le 1$

d. $-1 < r \le 1$

e. All values of $r$.

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & -1 < r < 1 \\ \text{DNE} & r \geq -1 \end{cases}$$

2. Determine whether the sequence converges or diverges. If it converges, find the limit.

a. $a_n = \frac{3n+1}{\sqrt{n+1}}$ $\lim_{n\to\infty} \frac{3n+1}{\sqrt{n+1}} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{3+1/n}{\sqrt{1/n + 1/n^2}} = \infty$ Diverges

b. $b_n = \frac{\ln(n)}{n}$ $\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{\ln(x)}{x} \ \infty/\infty = \lim_{x\to\infty} \frac{1}{x} = 0$

c. $c_n = \frac{\tan^{-1}(n)}{n+2}$ $\lim_{n\to\infty} \frac{\tan^{-1}(n)}{n+2} = 0$ b/c $\lim_{n\to\infty} \tan^{-1}(n) = \pi/2$

d. $d_n = e^{-2/\sqrt{n}}$ $\lim_{n\to\infty} e^{-2/\sqrt{n}} = e^{\lim_{n\to\infty} -2/\sqrt{n}} = e^0 = 1$

e. $e_n = \left(1 + \frac{2}{n}\right)^n$

$L = \lim_{n\to\infty} [1 + 2/n]^n \Rightarrow \ln L = \lim_{n\to\infty} n \cdot \ln(1 + 2/n)$ $\infty \cdot 0$

$= \lim_{x\to\infty} \frac{\ln(1+2/x)}{1/x}$ $0/0$

$= \lim_{x\to\infty} \frac{-2/x^2 \cdot \frac{1}{1+2/x}}{-1/x^2} = +2$

$\Rightarrow \ln L = 2$ so $\boxed{L = e^2}$

f. $f_n = \frac{4^n}{1+9^n}$

$\lim_{n\to\infty} \frac{4^n}{1+9^n} \cdot \frac{1/9^n}{1/9^n} = \lim_{n\to\infty} \frac{(4/9)^n}{1/9^n + 1}$ $\to 0$, $\to 0$ $= \frac{0}{1} = 0$

**Limit Laws for Sequences.** If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and if $c$ is a constant, then

- $\lim\limits_{n\to\infty}(a_n+b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty}(a_nb_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n$
- $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim\limits_{n\to\infty} b_n \neq 0$.

3. (Clicker) Assume $\lim\limits_{n\to\infty} a_n = 3$ and $\lim\limits_{n\to\infty} b_n = 5$. Compute $\lim\limits_{n\to\infty} \dfrac{a_n^2 - 2b_na_n}{a_n+b_n}$.

   a. 3

   b. $-\frac{31}{8}$

   c. $-\frac{21}{8}$

   d. 0

   e. $-\frac{3}{4}$

$$\lim_{n\to\infty}\frac{a_n^2-2b_na_n}{a_n+b_n} = \frac{9-2\cdot3\cdot5}{8} = \frac{9-30}{8} = \frac{-21}{8}$$

4. Use the Squeeze Theorem to determine the limit of $a_n = \dfrac{2^n}{n!}$.

5. Determine the limit of each of the following sequences or show that it diverges.

   a. $a_n = \sqrt{n+1}-\sqrt{n}$

$$\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = 0$$

   b. $b_n = \sqrt{n}\ln\left(1+\dfrac{1}{n}\right)$

$$\underset{0\cdot\infty}{\lim_{n\to\infty}\sqrt{n}\cdot\ln[1+1/n]} = \lim_{x\to\infty}\frac{\ln[1+1/x]}{x^{-1/2}} = \lim_{x\to\infty}\frac{-1/x^2\cdot\frac{1}{1+1/x}}{-1/2\,x^{-3/2}}$$
$$= 2\lim_{x\to\infty}\frac{1}{x^{1/2}[1+1/x]} = 0.$$

   c. $c_n = \ln\left(\dfrac{2n+1}{3n+4}\right)$

$$\lim_{n\to\infty}\ln\left[\frac{2n+1}{3n+4}\right] = \ln\left[\lim_{n\to\infty}\frac{2n+1}{3n+4}\right] = \ln[2/3]$$

   d. $d_n = n\sin\left(\dfrac{\pi}{n}\right)$

$$\underset{\infty\cdot 0}{\lim_{n\to\infty} n\cdot\sin\left(\frac{\pi}{n}\right)} = \lim_{x\to\infty}\frac{\sin(\pi/x)}{1/x}\ {}^{0/0} = \lim_{x\to\infty}\frac{-\pi/x^2\cos\left(\frac{\pi}{x}\right)}{-1/x^2} = \pi$$

   e. $e_n = \dfrac{e^n}{2^n}$

$$\lim_{n\to\infty}\frac{e^n}{2^n} = \lim_{n\to\infty}\left(\frac{e}{2}\right)^n = \infty \quad \text{since } \frac{e}{2} > 1$$