Solutions

Oct. 26th: 11.1: Sequences

Warm-Up

1 Find the next four terms of the following sequences:

   a. 1, 3, 5, 7, 9 ... 11, 13, 15, 17
   b. 1, -4, 9, -16, 25, ... -36, 49, -64, 81
   c. \( \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \cdots \) \( \frac{9}{17}, \frac{10}{19}, \frac{8}{21} \)

In-Class Exercises

Defn. A sequence is an infinite ordered list of numbers.

1. Find formulas for the \( n^{th} \) term of the following sequences:
   a. 1, 3, 5, 7, 9 ... \( a_n = 2n - 1 \)
   b. 1, -4, 9, -16, 25, ... \( b_n = (-1)^n \cdot n^2 \)
   c. \( \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \frac{7}{15}, \cdots \) \( c_n = \frac{n+1}{2n+3} \)

Defn: Sequence Vocabulary. Let \( \{a_n\} \) be a sequence.

   i. \( \{a_n\} \) is bounded above if there is some \( M \) such that \( a_n \leq M \) for all \( n \).
   ii. \( \{a_n\} \) is bounded below if there is some \( M \) such that \( a_n \geq M \) for all \( n \).
   iii. \( \{a_n\} \) is bounded if it is both bounded above and bounded below.
   iv. \( \{a_n\} \) is increasing if \( a_{n+1} \geq a_n \) for all \( n \).
   v. \( \{a_n\} \) is decreasing if \( a_{n+1} \leq a_n \) for all \( n \).
   vi. \( \{a_n\} \) is monotonic if it is either increasing or decreasing.

2. (Clicker) Consider the sequences given by the following formulas.
   \[ a_n = (-1)^n \cdot \frac{2n}{n+1}, \quad b_n = \frac{n^2}{(n+2)!}, \quad c_n = \frac{\sin(\pi n)}{\ln(n+1)}, \quad d_n = \frac{e^n}{n} \]

Which of those sequences are bounded?
   a. \( \{a_n\}, \{c_n\}, \) and \( \{d_n\} \)

   \[ |a_n| = \frac{2n}{n+1} \leq \frac{2n}{n} = 2 \quad \Rightarrow \quad -2 \leq a_n \leq 2 \]
   \[ 0 \leq b_n = \frac{n \cdot n}{(n+2)(n+1) \cdot \cdots \cdot 2 \cdot 1} \leq \frac{n}{n+2} \cdot \frac{n}{n+1} \leq 1 \]
   \[ |c_n| \leq \frac{1}{\ln(n+1)} \leq \frac{1}{\ln 2} = \frac{1}{\ln 2} \leq c_n \leq \frac{1}{\ln 2} \]
b. \(\{a_n\}\) and \(\{b_n\}\)

c. \(\{b_n\}\) and \(\{c_n\}\)

d. \(\{a_n\}, \{b_n\},\) and \(\{c_n\}\)

e. All of them

3. (Clicker) Consider the sequences given by the following formulas.

\[
\begin{align*}
    a_n &= (-1)^n \frac{2n}{n+1} \\
    b_n &= \frac{n^2}{(n+2)!} \\
    c_n &= \frac{\sin(\pi n)}{\ln(n+1)} \\
    d_n &= \frac{e^n}{n}
\end{align*}
\]

Which of those sequences are monotonic? \(\text{decreasing}\) \(\text{increasing}\)

a. \(\{a_n\}\), \(\{b_n\}\), and \(\{d_n\}\)

b. \(\{a_n\}\) and \(\{c_n\}\)

c. \(\{b_n\}\) and \(\{d_n\}\)

d. All of them

e. None of them

4. Determine whether the sequence converges or diverges. If it converges, find the limit.

a. \(a_n = \frac{1}{n^2} \quad \lim_{n \to \infty} \frac{1}{n^2} = 0\)

b. \(b_n = (-1)^n \frac{n+1}{n} \quad \text{some} \to 1 \quad \text{so diverges} \quad \text{some} \to -1\)

5. Determine whether the sequence converges or diverges. If it converges, find the limit.

a. \(a_n = \frac{3n+1}{\sqrt{n} + 1}\)

b. \(b_n = \frac{\ln(n)}{n}\)

c. \(c_n = \frac{\tan^{-1}(n)}{n+2}\)

d. \(d_n = e^{-2/\sqrt{n}}\)

e. \(e_n = \left(1 + \frac{2}{n}\right)^n\)

f. \(f_n = \frac{4^n}{1 + 9^n}\)