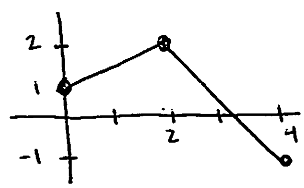
Solution

Oct. 19th: 8.1: Arc Length

Warm Up Exercises

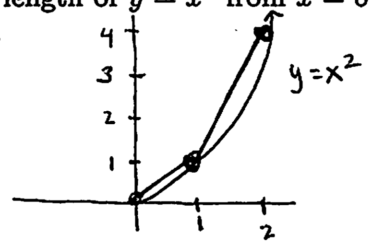
1. Compute the length of the curve draw below.



$\sqrt{(2-1)^2+(2-0)^2}+\sqrt{(4-2)^2+(-1-2)^2}$

$=\sqrt{5}+\sqrt{13}$

2. Compute the length of $y = x^2$ from $x = 0$ to $x = 2$ by computing the lengths of the approximating line segments.



$\sqrt{(1-0)^2+(1-0)^2}+\sqrt{(2-1)^2+(4-1)^2}$

$=\sqrt{2}+\sqrt{10}$

In-Class Exercises

1. (Clicker) Let $f(x) = \sqrt{4-x^2}$. Which of the following is a simplification of $\sqrt{1+[f'(x)]^2}$?

a. $\dfrac{4}{4-x^2}$

b. $\sqrt{4-x^2}$

c. $\dfrac{2}{\sqrt{4-x^2}}$

d. $\dfrac{\sqrt{4-x^2}}{2}$

$f'(x)=\dfrac{-x}{\sqrt{4-x^2}}$ so $\sqrt{1+[f'(x)]^2}=\sqrt{1+\dfrac{x^2}{4-x^2}}$

$=\dfrac{2}{\sqrt{4-x^2}}$

2. Compute the exact arc length of the following curves:

a. $y = \sqrt{4-x^2}$ from $x = 0$ to $x = 2$

$\text{arclength} = \int_0^2 \dfrac{2}{\sqrt{4-x^2}}\,dx = 2\arcsin(x/2)\Big|_0^2 = 2\cdot\dfrac{\pi}{2} = \pi$

Handwritten [b. $y = 2\ln\left(\sin\left(\frac{x}{2}\right)\right)$ for $\frac{\pi}{3} \le x \le \pi$

c. $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$

next pages, hand written

d. $y = \frac{x^4}{16} + \frac{1}{2x^2}$ for $1 \leq x \leq 2$

e. $y = \ln(1 - x^2)$ from $x = 0$ to $x = \frac{1}{2}$

[Clicker] Which of remaining examples do you want me to work? b, c, d, e.

b) $y = 2\ln(\sin(\frac{x}{2}))$ from $x = \pi/3$ to $x = \pi$

$$y' = \frac{2 \cdot \frac{1}{2}\cos(\frac{x}{2})}{\sin(x/2)} = \cot(\frac{x}{2})$$

$$\Rightarrow \sqrt{1+[f'(x)]^2} = \sqrt{1+\cot^2(x/2)} = \csc(x/2)$$

$$\text{arclength} = \int_{\pi/3}^{\pi} \sqrt{1+[f'(x)]^2}\,dx = \int_{\pi/3}^{\pi} \csc(x/2)\,dx = -\ln\left|\csc(\tfrac{x}{2}) + \cot(\tfrac{x}{2})\right|_{\frac{\pi}{3}}^{\pi}$$

$$= -\ln\left|\frac{1}{\sin(\pi/2)} + \frac{\cos(\pi/2)}{\sin(\pi/2)}\right| + \ln\left|\frac{1}{\sin(\pi/6)} + \frac{\cos(\pi/6)}{\sin(\pi/6)}\right|$$

$\sin\frac{\pi}{2} = 1$ $\quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{2} = 0$

$\sin\pi/6 = 1/2$

$$= -\ln(1) + \ln|2+\sqrt{3}|$$

$$\boxed{= \ln(2+\sqrt{3})}$$

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c) $y = \frac{1}{2}x^2 \quad 0 \leq x \leq 1$

$y' = x \qquad \sqrt{1+(y')^2} = \sqrt{1+x^2}$

$$\text{arclength} = \int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\pi/4} \sec\theta \cdot \sec^2\theta\,d\theta$$

use By-Parts

$u = \sec\theta \qquad dv = \sec^2\theta\,d\theta$

$du = \sec\theta\tan\theta\,d\theta \qquad v = \tan\theta$

$x = \tan\theta \qquad 0 = \tan\theta \Rightarrow \theta = 0$

$dx = \sec^2\theta\,d\theta \qquad 1 = \tan\theta \Rightarrow \theta = \pi/4$

$$= \sec\theta\tan\theta\Big|_0^{\pi/4} - \int_0^{\pi/4} \sec\theta \cdot \tan^2\theta\,d\theta \qquad \tan^2\theta = \sec^2\theta - 1$$

$$= \sqrt{2} - \int_0^{\pi/4} \sec^3\theta\,d\theta + \int_0^{\pi/4} \sec\theta\,d\theta$$

$$\Rightarrow 2\int_0^{\pi/4} \sec^3\theta\,d\theta = \sqrt{2} + \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4} = \sqrt{2} + \ln|\sqrt{2}+1|$$

$$\Rightarrow \boxed{\text{arclength} = \frac{1}{\sqrt{2}} + \frac{1}{2}\ln|\sqrt{2}+1|}$$

d) $y = \frac{x^4}{16} + \frac{1}{2x^2}$ For $1 \leq x \leq 2$

$$y' = \frac{4x^3}{16} - \frac{2x^{-3}}{2} = \frac{x^3}{4} - \frac{1}{x^3} \Rightarrow 1+[y']^2 = 1+\left[\frac{x^3}{4} - \frac{1}{x^3}\right]^2 = 1 + \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6}$$

$$= \left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2$$

$$\Rightarrow \text{arclength} = \int_1^2 \sqrt{1+[y']^2}\, dx = \int_1^2 \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2}\, dx$$

$$= \int_1^2 \frac{x^3}{4} + \frac{1}{x^3}\, dx = \frac{x^4}{16} - \frac{x^{-2}}{2} \Big|_1^2 = 1 - \frac{1}{8} - \left(\frac{1}{16} - \frac{1}{2}\right)$$

$$= \frac{3}{2} - \frac{3}{16} = \boxed{\frac{21}{16}}$$

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e) $y = \ln(1-x^2)$ $0 \leq x \leq 1/2$

$$y' = \frac{-2x}{(1-x^2)} \Rightarrow 1+[y']^2 = 1 + \frac{4x^2}{(1-x^2)^2} = \frac{1-2x^2+x^4+4x^2}{(1-x^2)^2} = \frac{(1+x^2)^2}{(1-x^2)^2}$$

$$\Rightarrow \text{arclength} = \int_0^{1/2} \sqrt{1+[y']^2}\, dy = \int_0^{1/2} \frac{1+x^2}{1-x^2}\, dx$$

$\therefore$ need partial Fractions

$$\begin{array}{r} -1 \ R2 \\ -x^2+1 \overline{) \, x^2+1} \\ -(x^2-1) \\ \hline 2 \end{array}$$

$$\Rightarrow \frac{1+x^2}{1-x^2} = -1 + \frac{2}{1-x^2} = -1 + \frac{A}{1-x} + \frac{B}{1+x} = -1 + \frac{1}{1-x} + \frac{1}{1+x}$$

$$\Rightarrow 2 = A(x+1) + B(1-x)$$

$x = 1 \Rightarrow 2 = 2A \Rightarrow A = 1$
$x = -1 \Rightarrow 2 = 2B \Rightarrow B = 1$

$$\Rightarrow \text{arclength} = \int_0^{1/2} -1 + \frac{1}{1-x} + \frac{1}{1+x}\, dx$$

$$= -x - \ln|1-x| + \ln|1+x| \Big|_0^{1/2}$$

$$= -1/2 - \ln(1/2) + \ln(3/2)$$

$$= -1/2 - \ln(1/2) + \ln(3) + \ln(1/2)$$

$$\boxed{= \ln 3 - 1/2}$$