Answer key

Oct. 5th: 7.4 Partial Fractions

Warm-Up

1 Use polynomial long division to write the following rational functions as $Q(x) + \frac{p(x)}{q(x)}$, for polynomials Q, p, q with $\deg p < \deg q$.

a. $\dfrac{x^2+1}{x^2+x+2}$

$1 \quad R -1$

$x^2+x+2 \,\overline{)\, x^2+0x+1}$

$-(x^2+x+2)$

-1

$$\boxed{1 + \frac{-1-x}{x^2+x+2}}$$

b. $\dfrac{2x^3+3x^2+4x+1}{x^2+1}$

$2x+3 \quad R\, 2x-2$

$x^2+1 \,\overline{)\, 2x^3+3x^2+4x+1}$

$-(2x^3 \quad +2x)$

$3x^2+2x+1$

$-(3x^2 \quad +3)$

$2x-2$

$$\boxed{2x+3 + \frac{2x-2}{x^2+1}}$$

c. $\dfrac{3x^4+3x^3+2x^2-x-3}{x+3}$

$3x^3-6x^2+20x-61 \quad R\, 180$

$x+3 \,\overline{)\, 3x^4+3x^3+2x^2-x-3}$

$-(3x^4+9x^3)$

$-6x^3+2x^2$

$-(-6x^3-18x^2)$

$20x^2-x$

$-(20x^2+60x)$

$-61x-3$

$-61x-3$

$-(-61x-61\cdot 3)$

180

$$\boxed{3x^3-6x^2+20x-61 + \frac{180}{x+3}}$$

InClass Exercises

1. (Clicker) What is the correct form of the partial fraction decomposition of $\dfrac{x^2+2x+3}{(x^2-4)\,(x-1)^3(x+3)}$?

a. $\dfrac{A_1+A_2x}{x^2-4} + \dfrac{A_3}{x-1} + \dfrac{A_4}{(x-1)^2} + \dfrac{A_5}{(x-1)^3} + \dfrac{A_6}{x+3}$

b. $\dfrac{A_1}{x-2} + \dfrac{A_2}{x+2} + \dfrac{A_3+A_4x+A_5x^2}{(x-1)^3} + \dfrac{A_6}{x+3}$

[✓ c.] $\dfrac{A_1}{x-2} + \dfrac{A_2}{x+2} + \dfrac{A_3}{x-1} + \dfrac{A_4}{(x-1)^2} + \dfrac{A_5}{(x-1)^3} + \dfrac{A_6}{x+3}$

d. $\dfrac{A_1}{x^2-4} + + \dfrac{A_2}{x-1} + \dfrac{A_3}{(x-1)^2} + \dfrac{A_5}{(x-1)^3} + \dfrac{A_6}{x+3}$

2. Find the partial fraction decompositions of the following rational functions.

a. $\dfrac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = 3x^2+1 + \dfrac{1}{x^2+x-2}$ (long division from last class)

$$\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$1 = A(x-1)+B(x+2)$

$x=1 \quad 1=3B \quad B=1/3$

$x=-2 \quad 1=-3A \quad A=-1/3$

$$\boxed{3x^2+1 \quad \frac{-1/3}{x+2} + \frac{1/3}{x-1}}$$

b. $\frac{x^2-4x+8}{(x-1)^2(x-2)^2}$ $= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)} + \frac{D}{(x-2)^2} = \frac{8}{x-1} + \frac{5}{(x-1)^2} + \frac{-8}{(x-2)} + \frac{4}{(x-2)^2}$

$\Rightarrow x^2-4x+8 = (x-2)^2(x-1)A + B(x-2)^2 + C(x-1)^2(x-2) + D(x-1)^2$

$x=1 \Rightarrow \quad 1-4+8 = B \quad B=5 \qquad x=2 \quad 4-8+8=D \quad D=4$

$x^3 \quad 0 = A+C \qquad\qquad A+C=0 \qquad A=8$

$1 \quad 8 = -4A+4B-2C+D = -4A+20+4-2C \Rightarrow -2A-C=-8 \qquad C=-8$

c. $\frac{x^2}{x^2-9}$

$1 \text{ R}9$

$x^2-9 \overline{)\, x^2+0x+0}$

$-(x^2-9)$

9

$= 1 + \frac{1}{x^2-9} = 1 + \frac{1}{(x-3)(x+3)}$

$= 1 + \frac{A}{x-3} + \frac{B}{x+3}$

$1 = A(x+3) + B(x-3)$

$= 1 + \frac{1/6}{x-3} - \frac{1/6}{x+3}$

$x=-3 \quad 1=-6B \quad B=-1/6$

$x=3 \quad 1=6A \quad A=1/6$

3. Compute the following integrals:

a. $\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$ $= \int 3x^2+1 + \frac{-1/3}{x+2} + \frac{1/3}{x-1}\,dx$

$= x^3 + x - 1/3 \ln|x+2| + 1/3 \ln|x-1| + C$

b. $\int \frac{x^2-4x+8}{(x-1)^2(x-2)^2}\,dx$ $= \int \frac{8}{x-1} + \frac{5}{(x-1)^2} - \frac{8}{x-2} + \frac{4}{(x-2)^2}\,dx$

$= 8\ln|x-1| - 5/x-1 - 8\ln|x-2) - 4/x-2 + C$

c. $\int \frac{x^2}{x^2-9}\,dx$ $= \int 1 + \frac{1/6}{x-3} - \frac{1/6}{x+3}\,dx = x + 1/6 \ln|x-3| - 1/6 \ln|x+3| + C$

4. (Clicker) Find the form of the partial fraction decomposition of $\frac{x^3+3x^2-10x+25}{(x^2-1)^2(x^2+1)^2}$

next class

a. $\frac{A+Bx}{x^2-1} + \frac{C+Dx}{(x^2-1)^2} + \frac{E+Fx}{x^2+1} + \frac{G+Hx}{(x^2+1)^2}$

b. $\frac{A}{x-1} + \frac{B}{x+1} + \frac{C+Dx}{(x^2-1)^2} + \frac{E+Fx}{x^2+1} + \frac{G+Hx}{(x^2+1)^2}$

c. $\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x-1)^2} + \frac{D}{(x+1)^2} + \frac{E}{x^2+1} + \frac{G}{(x^2+1)^2}$

ⓓ $\frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x-1)^2} + \frac{D}{(x+1)^2} + \frac{E+Fx}{x^2+1} + \frac{G+Hx}{(x^2+1)^2}$