Answer Key

# September 2nd: 5.3: The Fundamental Theorem of Calculus

## Extra Practice Exercise

1. Compute $\int_0^4 (x+3)\ dx$ using the limit definition of the definite integral, i.e. write:

$$\int_0^4 (x+3)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

and compute the right-hand-side by following these steps:

a. Break $[0,4]$ into $n$ subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ of equal length $\Delta x$. Write down a formula for $\Delta x$. Write down a formula for $x_i$ that works for $i = 1, 2, \ldots, n$.

$\Delta x = \frac{4}{n}$ $\quad x_i = a + i\Delta x = \frac{4i}{n}$

b. Select sample points $x_1^*, \ldots, x_n^*$, one coming from each of the $n$ subintervals. Specifically, for $i = 1, 2, \ldots, n$ choose $x_i^* = x_i$ from (a) and then write down a formula for $f(x_i^*)$.

$x_i^* = \frac{4i}{n}$ $\quad f(x_i^*) = \frac{4i}{n} + 3$

c. Plug your answers from (a) and (b) into the sum: $\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} (4i/n + 3)\cdot \frac{4}{n}$

d. Using algebra and the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, simplify your formula from (c) so that it no longer contains a $\sum_{i=1}^{n}$.

$$\sum_{i=1}^{n}\left[\frac{4i}{n}+3\right]\cdot\frac{4}{n} = \frac{16}{n^2}\cdot\frac{n(n+1)}{2} + 12 = \frac{8(n+1)}{n} + 12$$

e. Compute $\int_0^4 (x+3)\ dx$ by computing $\lim_{n\to\infty} \sum_{i-1}^{n} f(x_i^*)\Delta x$ using your formula from (d).

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \frac{8(n+1)}{n} + 12 = 8 + 12 = 20$$

## Warm-Up Exercise

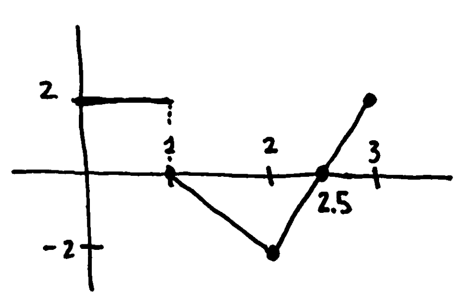
2. (Clicker) Define $G(x) = \int_0^x f(t)\ dt$, where $y = f(t)$ is graphed below. Using the area interpretation of the definite integral, compute $G(0)$, $G(1)$, $G(2)$, $G(3)$. Which is largest?

a. $G(0) = 0$

b. $G(1) = 2$

c. $G(2) = 2 - 1 = 1$

d. $G(3) = 1 - \frac{1}{2} + \frac{1}{2} = 1$



**Fundamental Theorem of Calculus (FTC).** Let $f$ be continuous on $[a, b]$. Then:

- Part 1: If $G(x) = \int_a^x f(t)\,dt$, then $G$ is an antiderivative of $f$ (i.e. $G'(x) = f(x)$).
- Part 2: If $F$ is any antiderivative of $f$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

**Class-time Exercises**

1. Compute the derivatives of the following functions using the FTC Part 1.

   a. $G(x) = \int_0^x (t^5 - 9t^3)\,dt$ $\quad G'(x) = x^5 - 9x^3$

   b. $F(x) = \int_{x^2}^{x^4} \sqrt{t}\,dt$ $= -\int_0^{x^2} \sqrt{t}\,dt + \int_0^{x^4} \sqrt{t}\,dt \quad F'(x) = -2x\sqrt{x^2} + 3x^4\sqrt{x^4}$

   c. $H(x) = \int_{-6}^{\cos x} t^4\,dt$ $\quad H'(x) = (-\sin x)(\cos x)^4$

   d. $I(x) = \int_{\sqrt{x}}^{x^2} \tan t\,dt$ $= -\int_0^{\sqrt{x}} \tan t\,dt + \int_0^{x^2} \tan t\,dt \quad I'(x) = -\frac{1}{2}\cdot x^{-1/2}\tan(\sqrt{x}) + 2x\tan(x^2)$

2. (Clicker) Which of the following is an antiderivative of $\ln x$?

   a. $G(x) = \int_x^2 \ln x\,dx$

   b. $F(x) = x\ln x + x$

   c. Neither of the above

   d. Both of the above