Partial Fractions
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The point of partial fractions is to turn a rational function (a quotient of polynomials) into simpler fractions which are easier to integrate. Partial fraction is an *algebraic* method that we apply to integrals. Here are the main steps. First, we start with a rational function that we want to integrate:

$$\int \frac{P(x)}{Q(x)}\, dx$$

where $P(x)$ and $Q(x)$ are polynomials.

1. First we want to make sure that the degree of $P(x)$ is *less then* the degree of $Q(x)$. IF this is not the case, we use long division.

2. Next, after long dividing (if it was necessary), we factor $Q(x)$ completely into its linear and quadratic factors. This can be difficult but most of your calculators have polynomial solvers that can help you do this.

3. Now, we determine the "form" that the partial fraction decomposition should take. This depends on the factors that you found for $Q(x)$. Here are the different possibilities (I'll put some examples down later):

   (a) $Q(x)$ has a linear factor (something of the form $x - \alpha$. This linear factor "contributes" a fraction of the form $\frac{A}{x-\alpha}$ to the partial fraction decomposition.

   (b) $Q(x)$ has "repeated" linear factors. This means that $Q(x)$ has a factor that looks like $(x - \alpha)^3$ (i.e., the factor is linear, but it is repeated more then once, in this case it was repeated three times). In this case, the factor $(x - \alpha)^3$ contributes the following to the partial fraction decomposition: $\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2} + \frac{C}{(x-\alpha)^3}$.

   (c) $Q(x)$ has an irreducible quadratic factor: $ax^2 + bx + c$. A factor of this form contributes the following to the partial fraction decomposition: $\frac{Ax+B}{ax^2+bx+c}$.

   (d) If the quadratic factor is repeated, then we do a something similar to what we did for the repeated linear factor. Rather then give a rule, see what we do in the examples.

   Here are some examples of setting up the correct form (notice that the degree of $P(x)$ is strictly less then the degree of $Q(x)$):

$$\frac{2x}{x^2 - 3x + 2} = \frac{2x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$\frac{x}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$$

$$\frac{1}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\frac{1}{(x-1)^2(x-2)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} + \frac{E}{(x-2)^3}$$

$$\frac{2x}{(x^2+x+1)(x-1)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}$$

$$\frac{2x}{(x^2+x+1)^2(x-1)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{E}{x-1}$$

4. Once you have found the correct form, the idea now is the solve for your unknowns (the A, B, C, etc.).

5. Once you have solved for the unknowns, the resulting integral is easier to compute (although not always simple, it is better then the original integral).

Extra Problems using Partial Fractions

1. Write out the "form" for the partial fraction decomposition in each case below. (More work: find the constants and find the corresponding integral. Do this at home.)

   (a) $\frac{2x+3}{(x+1)(2x-4)}$

   (b) $\frac{2x+3}{(x^2+4x+3)(x-1)}$

   (c) $\frac{2x+3}{(x+1)^2(x-1)(x^2+x+1)}$

   (d) $\frac{2x+2}{(x^2+x+1)^2(x-1)^2(x^2-x+1)}$

   (e) $\frac{2x^2+x+1}{x^2+2x-3}$

2. Find the integrals:

   (a) $\int \frac{x}{(x-1)^2(x+3)}\, dx$
   Solution: the partial fraction decomposition should be: $\frac{3}{16}\frac{1}{(x-1)} + \frac{1}{4}\frac{1}{(x-1)^2} - \frac{3}{16}\frac{1}{(x+3)}$

   (b) $\int \frac{x-1}{(x+1)(2x^2+x+1)}\, dx$
   Solution: the partial fraction decomposition: $\frac{-1}{x+1} + \frac{2x}{2x^2+x+1}$