Warm-Up Problems and Lecture Problems
Solutions
March 21, 2003

1. As you come in, please tell me your height in inches.

2. Consider the function $f(t) = Ae^{-ct}$ for $t \geq 0$ where A and c are constants.

   (a) Suppose you want to ensure $\int_0^\infty f(t)\,dt = 1$. Find conditions on A and c to ensure this.
   Solution: We just compute

   $$\begin{aligned}\int_0^\infty Ae^{-ct}\,dt &= \lim_{u\to\infty}\int_0^u Ae^{-ct}\,dt = \lim_{u\to\infty} -\frac{A}{c}e^{-ct}\Big|_0^u = \lim_{u\to\infty} -\frac{A}{c}(e^{-cu} - e^{-0c}) \\ &= \lim_{u\to\infty} -\frac{A}{c}(e^{cu} - 1) = \frac{A}{c}\end{aligned}$$

   So, we need $\frac{A}{c} = 1$ or $A = c$.

   (b) Suppose, in addition, you know that $A = 0.01$, what is c?
   Solution: According to the previous work, $c = 0.01$.

   (c) Using your $A = 0.01$ and the c that you just found, find a nice expression for the function:

   $$F(t) = \int_0^t f(u)\,du$$

   Solution: Just do the integration:

   $$F(t) = \int_0^t (0.01)e^{-(0.01)u}\,du = -e^{-(0.01)u}\Big|_0^t = 1 - e^{-(0.01)t}$$

   (d) What is $\lim_{t\to\infty} F(t)$?
   Solution: Just take the limit:

   $$\lim_{t\to\infty} F(t) = 1$$

   (e) Compute: $\int_0^\infty tf(t)\,dt$.
   Solution: This is integration by parts:

   $$\int_0^\infty t(0.01)e^{-0.01t}\,dt$$

   Let $u = 0.01t$ and $dv = e^{-0.01t}\,dt$. This gives $du = 0.01\,dt$ and $v = -100e^{-0.01t}$.

   $$\begin{aligned}uv - \int v\,du &= -te^{-0.01t} + \int e^{-0.01t}\,dt \\ &= -te^{-0.01t} - 100e^{-0.01t} + C\end{aligned}$$

   We can take limits and plug in our endpoints:

   $$\begin{aligned}\int_0^\infty t(0.01)e^{-0.01t}\,dt &= \lim_{u\to\infty}\int_0^u t(0.01)e^{-0.01t}\,dt = \lim_{u\to\infty} -te^{-0.01t} - 100e^{-0.01t}\Big|_0^u \\ &= \lim_{u\to\infty}(-ue^{-0.01u} - 100e^{-0.01u}) - (-100) = 100\end{aligned}$$