

Strategies for Integration

Up to this point in the course, we have been presented with several integration techniques spanning several sections of chapter 5. For example, integration by parts was discussed in section 5.6. However, in practice, we will not know ahead of time which method is applicable. Instead, we will need to learn how to recognize which technique or formula to use. The following is a list of strategies to help us learn to recognize which technique to apply.

(i) **Simplify the integrand if possible.** Sometimes algebraic simplification or use of trigonometric identities will simplify the integrand and make the method of integration apparent. For example,

$$\int \frac{\tan \theta}{\sec^2 \theta} d\theta = \int \frac{\sin \theta}{\cos \theta} \cos^2 \theta d\theta = \int \sin \theta \cos \theta d\theta$$

or

$$\int x^3(x^2 + \sqrt{x} + 1) dx = \int (x^5 + x^{7/2} + x^3) dx.$$

(ii) **Look for an obvious substitution.** For example, letting $u = \sin \theta$ in the example above turns

$$\int \sin \theta \cos \theta d\theta \text{ into } \int u du,$$

which can be easily integrated.

(iii) **Classify the integrand according to its form.** Let's call the integrand $f(x)$.

(a) **Integration by Parts.** If $f(x)$ is a product of a polynomial and a transcendental function (such as a trigonometric, exponential, or logarithmic function), then try integration by parts. Remember that the object in using parts is to obtain a simpler integral than the one with which we started, so make your choice of u and dv accordingly. A helpful device in selecting u is to use the mnemonic LIATE (for Long Island ATE). L stands for logarithmic, I for inverse trigonometric, A for algebraic, T for trigonometric, and E for exponential. Try to choose for u the factor that appears highest on the list.

(b) **Trigonometric Integrals** If $f(x)$ is a product of powers of sine and cosine, then try one of the following substitutions:

(1) If the power of cosine is odd, save one factor of cosine and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine. For example,

$$\int \sin^2 x \cos^3 x dx = \int \sin^2 x \cos^2 x \cos x dx = \int \sin^2 x (1 - \sin^2 x) \cos x dx.$$

Then use the substitution $u = \sin x$.

(2) If the power of sine is odd, save one factor of sine and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine. For example,

$$\int \sin^5 x \cos^2 x dx = \int \sin^4 x \cos^2 x \sin x dx = \int (1 - \cos^2 x)^2 \cos^2 x \sin x dx.$$

Then use the substitution $u = \cos x$.

(c) **Trigonometric Substitutions.** If $\sqrt{\pm x^2 \pm a^2}$ occurs, then we try one of the following trigonometric substitutions:

$$\begin{array}{ll} \sqrt{a^2 - x^2}, & \text{let } x = a \sin \theta, \text{ then use the identity } 1 - \sin^2 \theta = \cos^2 \theta \\ \sqrt{a^2 + x^2}, & \text{let } x = a \tan \theta, \text{ then use the identity } 1 + \tan^2 \theta = \sec^2 \theta \\ \sqrt{x^2 - a^2}, & \text{let } x = a \sec \theta, \text{ then use the identity } \sec^2 \theta - 1 = \tan^2 \theta \end{array}$$

The result is an integral involving trig functions.

(d) **Partial Fractions.** If $f(x)$ is a rational function (i.e. a ratio of polynomials), we express it as a sum of simpler fractions that are easier to integrate. Suppose we want to integrate $\int \frac{P(x)}{Q(x)} dx$, where P and Q are polynomials.

- (1) We must have $\text{degree}(P(x)) < \text{degree}(Q(x))$. If that is not the case with the given integrand, then perform “long division” to get such a rational function.
- (2) Factor the denominator $Q(x)$ into linear and “irreducible” (i.e. unfactorable) quadratic factors.
- (3) Write the expression representing $\frac{P(x)}{Q(x)}$ as a sum of simpler fractions. We will call this expression the “partial fraction decomposition” or “pfd” for $\frac{P(x)}{Q(x)}$.
 - (i) If there are only linear factors, and all are distinct, then each factor $(x - \alpha)$ “contributes” a fraction like $\frac{A}{x - \alpha}$ to the pfd. For example,

$$\frac{4x}{x^2 - x - 2} = \frac{4x}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}.$$

- (ii) A repeated linear factor contributes “extra” fractions to the pfd. For example,

$$\frac{1}{(x+1)^2(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-2)}.$$

- (iii) Each distinct, irreducible, quadratic factor $ax^2 + bx + c$ of $Q(x)$ contributes a fraction $\frac{Ax+B}{ax^2+bx+c}$ to the pfd. For example,

$$\frac{4x}{(x^2+x+1)(x-1)} = \frac{Ax+B}{(x^2+x+1)} + \frac{C}{(x-1)}.$$

- (iv) Each repeated quadratic factor contributes “extra” terms to the pfd. For example,

$$\frac{4x+1}{(x^2+x+1)^2(x+1)^2(x-1)} = \frac{Ax+B}{(x^2+x+1)} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{E}{(x+1)} + \frac{F}{(x+1)^2} + \frac{G}{(x-1)}.$$

(iv) **Use several methods.** Sometimes more than one method is required to evaluate an integral. We may have to make a substitution first and then use a second technique. For example, to integrate $\int e^{\sqrt{x}} dx$, we make the substitution $u = \sqrt{x}$. Then $x = u^2$ and $dx = 2u du$, so

$$\int e^{\sqrt{x}} dx = 2 \int u e^u du,$$

and we use integration by parts to evaluate the integral.

- (v) **Symmetry.** If $f(x)$ is odd, then $\int_{-a}^a f(x) dx = 0$. If $f(x)$ is even, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.