Warm-Up Problems and Lecture Problems
Solutions
February 7, 2003

1. For each of the following integrals, determine the correct choice of trigonometric substitution. (Use the chart handed out last time.)

   (a)
   $$\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$$

   Solution: $x = 5\sec\theta$

   (b)
   $$\int \frac{1}{(25+16x^2)^{3/2}}\,dx$$

   Solution: $x = \frac{5}{4}\tan\theta$

   (c)
   $$\int \frac{\sqrt{5-x^2}}{x}\,dx$$

   Solution: $x = \sqrt{5}\sin\theta$

2. For the substitutions made in the previous problem, convert the following expressions into expressions of x. For example, if your substitution was $x = \sin\theta$ and you are asked to put $\sin^2\theta$ into an expression of x, you would write $\sin^2\theta = x^2$.

   Remember to draw the appropriate triangle!

   (a) $\theta + \cos\theta - \tan\theta$
   Solution: $\cos^{-1}(5/x) + (5/x) - \frac{1}{5}\sqrt{x^2-25}$

   (b) $\theta + \cos\theta - \tan\theta$
   Solution: $\tan^{-1}(4x/5) + \frac{5}{\sqrt{25+16x^2}} - \frac{4x}{5}$

   (c) $\theta + \cos\theta - \tan\theta$
   Solution: $\sin^{-1}(x/\sqrt{5}) + \frac{1}{\sqrt{5}}\sqrt{5-x^2} - \frac{5}{\sqrt{5-x^2}}$

3. Write the following fractions in the correct "partial fraction decomposition form:"

(a)

$$\frac{x^2+3x}{(x^2+x+1)(x-2)}$$

Solution:

$$\frac{x^2+3x}{(x^2+x+1)(x-2)} = \frac{Ax+B}{(x^2+x+1)} + \frac{C}{(x-2)}$$

(b)

$$\frac{5x^3-1}{(2x+3)^3(x+4)}$$

Solution:

$$\frac{A}{(2x+3)} + \frac{B}{(2x+3)^2} + \frac{C}{(2x+3)^3} + \frac{D}{(x+4)}$$

4. Solve for the constants in the following partial fraction decomposition:

$$\frac{5}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2}$$

Solution: : "Clear" the denominators:

$$5 = A(x-2) + B(2x+1)$$

Now collect powers of x:

$$5 = (A+2B)x + (-2A+B)$$

Now set up equations to solve:

$$\begin{aligned} A + 2B &= 0 \\ -2A + B &= 5 \end{aligned}$$

Now solve the equations and get: $A = -2$ and $B = 1$

5. Here is an integral that I have done the partial fraction decomposition. Do the rest of the work and compute the antiderivative. (You might want to perform the partial fraction decomposition at home later.)

$$\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\, dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3x - 2}{x^2 + 1}\right)\, dx$$

Solution: The main idea is to split up the integral:

$$\begin{aligned}
\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\, dx &= \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3x - 2}{x^2 + 1}\right)\, dx \\
&= \int \frac{2}{x}\, dx - \int \frac{1}{x^2}\, dx + \int \frac{3x - 2}{x^2 + 1}\, dx \\
&= 2\ln|x| + \frac{1}{x} + 3\int \frac{x}{x^2 + 1}\, dx - 2\int \frac{1}{x^2 + 1}\, dx
\end{aligned}$$

We now use a u-substitution ($u = x^2 + 1$) to do the second to last integral and the last integral is $\tan^{-1} x$:

$$\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\, dx = 2\ln|x| + \frac{1}{x} + \frac{3}{2}\ln(x^2 + 1) - 2\tan^{-1} x + C$$