Warm-Up Problems and Lecture Problems
Solutions
February 24, 2003

1. Using Euler's method, with step size 0.5, estimate $y(3)$ for the initial value problem:

$$\frac{dy}{dx} = y - xy \qquad y(2) = 4$$

Solution:

n	x_n	y_n
0	2.0	4
1	2.5	2.0
2	3.0	0.5

2. Consider the differential equation:

$$\frac{dy}{dx} = \frac{x^7 - 12\sin^5 x}{x^4 + 1}$$

Is the following function a solution to this differential equation? Why or why not?

$$F(x) = \int_0^x \frac{t^7 - 12\sin^5 t}{t^4 + 1}\, dt$$

Solution: Yes, $F(x)$ is a solution. This is a review of the Fundamental Theorem of Calculus (part 1) – do you remember how to take the derivative of an integral function?

Lecture Problems

3. Determine if the following differential equations are separable. After making this determination for all the equations, solve the separable equations.

   (a) $\frac{dy}{dx} = 2x\sqrt{y}$
   Solution: Separate: $y^{-1/2}\,dy = 2x\,dx$, then integrate to get: $2\sqrt{y} = x^2 + C$. If you want, you can solve for y to get $y = \frac{1}{4}(x^2 + C)^2$

   (b) $\frac{dy}{dx} - 2y = 3e^{2x}$
   Solution: Not separable.

   (c) $x\frac{dy}{dx} + 3y = 2x^5 \quad y(2) = 1$
   Solution: Not separable.

   (d) $\frac{dy}{dx} = \frac{1}{4y^3} \quad y(0) = 1$
   Solution: Separate: $4y^3\,dy = dx$ and then integrate: $y^4 = x + C$. Use the initial condition to get $C = 1$ and the solution: $y^4 = x + 1$.

   (e) $\frac{dy}{dx} = 3x^2y^2 - y^2 \quad y(0) = 1$
   Solution: First factor: $\frac{dy}{dx} = (3x^2 - 1)y^2$. Now separate: $y^{-2}\,dy = (3x^2 - 1)\,dx$. Integrate: $-\frac{1}{y} = x^3 - x + C$. Solve for y to get: $y = \frac{-1}{x^3 - x + C}$. Use the initial condition to get $C = -1$ and the solution is $y = \frac{-1}{x^3 - x - 1}$.

   (f) $\frac{dy}{dx} + 2xy = x$
   Solution: Rearrange to get $\frac{dy}{dx} = x(1 - 2y)$ and then separate:' $\frac{dy}{1-2y} = x\,dx$. Integrate to get $-\frac{1}{2}\ln|1 - 2y| = \frac{x^2}{2} + C$. You can rearrange to get either $\ln|1 - 2y| = -x^2 - 2C$ or $y = Ae^{-x^2} + \frac{1}{2}$ (where $A = -\frac{1}{2}e^{-2C}$).

4. Find a family of curves orthogonal to the family $y = \frac{k}{x}$.
   Solution: First take the derivative: $\frac{dy}{dx} = -\frac{k}{x^2}$. We now solve for k in the original family to get $k = xy$ and substitute this into the DE. This gives $\frac{dy}{dx} = -\frac{y}{x}$. This is the differential equation representing the *given* family. The differential equation representing the orthogonal family will be given by the negative reciprocal of this one. In other words, we need to solve $\frac{dy}{dx} = \frac{x}{y}$, which has the solution:

   $$y^2 - x^2 = C$$

   (you might have $y^2 - x^2 = 2C$). Notice that these are two families of orthogonal hyperbolas.