Warm-Up Problems and Lecture Problems
Solutions
April 4, 2003

Series Toolbox: (Look up the precise details in the textbook – some of the statements below aren't precisely correct.)

I. **Geometric series:** $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.

II. **Divergence test:** If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

III. **Be careful:** If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ might diverge or it might converge – you have to do more work!

IV. **Integral test:** (Use this if the integral looks easy or at least doable.) Suppose $f(n) = a_n$. $f(x) \geq 0$ continuous, decreasing.

   (a) If $\int_1^{\infty} f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

   (b) If $\int_1^{\infty} f(x)\,dx$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

V. **p-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. (This can be shown using the integral test!)

VI. **Comparison test:** (Use this if the given series looks close to a series you know.) Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

   (a) If $\sum b_n$ converges and $a_n \leq b_n$ then $\sum a_n$ converges.

   (b) If $\sum b_n$ diverges and $a_n \geq b_n$ then $\sum a_n$ diverges.

VII. **Limit comparison test:** (Use this for comparisons that "don't quite work.") Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ with $0 < c < \infty$. Then, $\sum a_n$ and $\sum b_n$ either both converge or both diverge (they behave the same way).

VIII. **Telescoping series:** You need to use partial fractions for these. See problem 1 below.

1. Does the following series converge or diverge. If it converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

Solution: Do partial fractions:

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

If you now perform the infinite sum, the terms all cancel and you will be left with:

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{1}{2}$$

2. Converge or diverge

   (a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
   Solution: Converges (use the integral test).

   (b) $\sum_{n=1}^{\infty} ne^{-n^2}$
   Solution: Converges (integral test).

3. A geometric series has sum 5 and ratio $\frac{1}{3}$. What is the first term of the series?
   Solution: You know that the sum is equal to $\frac{a}{1-r}$. So you solve the equation: $5 = \frac{a}{1-(1/3)}$ to get $a = \frac{10}{3}$.

Lecture Problems

4. For each of the following series, find a good series to use in comparison:

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^3+5n}$
   Solution: Compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$

   (b) $\sum_{n=3}^{\infty} \frac{1}{n^3-5n}$
   Solution: Compare to $\sum_{n=3}^{\infty} \frac{1}{n^3}$

   (c) $\sum_{n=1}^{\infty} \frac{5^n+1}{2^n-1}$
   Solution: Compare to $\sum_{n=1}^{\infty} \frac{5^n}{2^n}$

   (d) $\sum_{n=1}^{\infty} \frac{5^n-1}{2^n+1}$
   Solution: Compare to $\sum_{n=1}^{\infty} \frac{5^n}{2^n}$

   (e) $\sum_{n=1}^{\infty} \frac{n^2-2n+5}{n^5+5n^2-5n+11}$
   Solution: Compare to $\sum_{n=1}^{\infty} \frac{n^2}{n^5}$

5. For each of the series in problem 4, try to set up the appropriate inequality between the terms in the given series and the series you want to compare to. Make sure your inequality is correct. (There might be some that just don't work.)
   Solution:

   (a) $\frac{1}{n^3+5n} \leq \frac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, this comparison shows that our series converges too.

   (b) $\frac{1}{n^3-5n} \geq \frac{1}{n^3}$. This comparison doesn't let us use the comparison test. We will have to use the limit comparison test.

   (c) $\frac{5^n+1}{2^n-1} \geq \frac{5^n}{2^n}$. Since $\sum_{n=1}^{\infty} \frac{5^n}{2^n}$ diverges, this comparison shows that our series diverges.

   (d) $\frac{5^n-1}{2^n+1} \leq \frac{5^n}{2^n}$. This comparison doesn't allow the use of the comparison test. We will have to use the limit comparison test.

   (e) $\frac{n^2-2n+5}{n^5+5n^2-5n+11}$ is a mess! We'll use the limit comparison test.

6. For the series in problem 4 use the limit comparison test to check convergence or divergence.
   Solution:

   (a)
   $$\lim_{n\to\infty} \frac{\frac{1}{n^3+5n}}{\frac{1}{n^3}} = 1$$

   (b)
   $$\lim_{n\to\infty} \frac{\frac{1}{n^3-5n}}{\frac{1}{n^3}} = 1$$

   (c)
   $$\lim_{n\to\infty} \frac{\frac{5^n+1}{2^n-1}}{\frac{5^n}{2^n}} = 1$$

   (d)
   $$\lim_{n\to\infty} \frac{\frac{5^n-1}{2^n+1}}{\frac{5^n}{2^n}} = 1$$

   (e)
   $$\lim_{n\to\infty} \frac{\frac{n^2-2n+5}{n^5+5n^2-5n+11}}{\frac{n^2}{n^5}} = 1$$

7. Which of the series in problem 4 diverge or converge.
   Solution:

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^3+5n}$ converges
   (b) $\sum_{n=3}^{\infty} \frac{1}{n^3-5n}$ converges
   (c) $\sum_{n=1}^{\infty} \frac{5^n+1}{2^n-1}$ diverges
   (d) $\sum_{n=1}^{\infty} \frac{5^n-1}{2^n+1}$ diverges
   (e) $\sum_{n=1}^{\infty} \frac{n^2-2n+5}{n^5+5n^2-5n+11}$ converges